Abstract— Joining two or more curve segments together forms a continuous composite curve. We start the discussion of composite curves and their continuity by investigating how to blend a new curve between two or more existing curves, forming a composite curve consisting of two or more curve segments together. In geometric modeling, curves are joined with smooth transitions at the junction. When joining two or more curves, the end points must coincide. In order to maintain smoothness, their first \( n \) parametric derivatives must be equal at that point and this will create a condition of continuity called \( n^{th} \)-order parametric continuity. In this paper conditions for smooth connection of interval Bezier curve segments are presented. The conditions for joining two or more interval Bezier curve segments at a common point are converted into the conditions for smooth connections of the four fixed Kharitonov's polynomials (four fixed Bezier curves) associated with the first interval Bezier curve with the corresponding four fixed Kharitonov’s polynomials (four fixed Bezier curves) associated with the second interval Bezier curve at common points \( \alpha^a_{j} = \beta^b_{j} \) for \( j = 1, 2, 3, 4 \). Finally, the interval control points of the second interval Bezier curve are obtained from conditions for smooth connections of the four fixed Kharitonov’s polynomials (four fixed Bezier curves) associated with the first interval Bezier curve with the corresponding four fixed Kharitonov's polynomials (four fixed Bezier curves) associated with the second interval Bezier curve. A numerical example is included in order to demonstrate the effectiveness of the proposed method.

Index Term— Smooth connection, CAD, interval Bezier curve, CAGD.

I. INTRODUCTION

CAD processes focus on the representation of product geometry and topology and include all required geometry modeling operations and structure-related organizational procedures. The geometrical representation of product models can require the creation of complex curves, surfaces and solids within a virtual environment (CAD-working space), whereby different technologies of geometry definition, approximation, interpolation and transformations come into use. Besides the creation of geometry models, CAD provides the basis for subsequently performed engineering tasks and model preparation. In computer-aided design, the product geometry is composed of several basis-elements (i.e., lines, curves, different types of surfaces and volume elements). Depending on the type of product and the desired characteristics of the virtual product model, these elements are combined by applying specific methods and strategies, which are called design rules. In general, complex CAD models consist of numerous components and modules, which are put together and structured in assemblies.

Some curve-defining techniques, such as those using the Hermite basis functions, interpolate a given set of points. This means that the curve produced passes exactly through those points. Other techniques define a curve that only passes near or approximates the given points. Interpolation techniques like the Hermite basis have certain disadvantages when incorporated into an interactive geometric-modeling system. They usually do not give the user an intuitive sense of how to change or control the shape of a curve. For example, changing the shape of a spline-interpolated curve by moving one or more of the interpolating points may produce unexpected, if not undesirable, perturbations and inflections, both locally and globally. To control curve shape in a predictable way by changing only a few simple parameters is more desirable. The Bezier curve partially satisfies this need. Bezier curves are established as a standard tool in computer aided geometric design. These polynomial curves possess a number of very desirable properties.

A new representation form of parametric curves, the interval Bezier curves, is introduced in [1], which is computed in interval arithmetic [2]. Interval arithmetic leads to numerical robustness and provides results with numerical certainty and verifiability. In [3] and [4] interval algorithms are applied to a wide variety of problems in computer graphics and geometry processing. In [5] interval arithmetic is used in non-linear polynomial system solvers. In [6] and [7] robust algorithms for curve and surface intersections by rounded interval arithmetic are presented. Interval arithmetic is also used in solid modeling [8], [9] and in a robust visualization [10]. In [11] two methods for performing robust rounded interval arithmetic are presented. The above series of works indicate that using rounded interval arithmetic will substantially provide results with verifiable numerical certainty in geometric computations, and thus enhance numerical robustness of current CAD/CAM systems.

Interval Bezier curves differ from classical Bezier curves in that the real numbers representing control point coordinates are replaced by intervals, such curves have been used in [12] for approximating offsets of parametric curves. In [13] interval Bezier curves are used for the representation of functions with uncertainty. Interval Bezier curves were used for solving shape interrogation problems robustly [14] and [15]. In [16] the problem of bounding interval Bezier curves with lower degree interval Bezier curves is discussed.

In interval Bezier curves, the classical control points are replaced by 2D rectangles or 3D boxes. Consequently, an interval curve represents a region containing a family of curves. This implies that, in 2D plane, an interval Bezier curve represents a thin strip and in 3D space, an interval
Bezier curve represents a slender tube, if the intervals are chosen sufficiently small.

Parametric interval Bezier curves are interactive. It is possible to control the shape of the interval Bezier curve by moving the interval control points and by smoothly connecting individual interval segments.

In this paper two interval Bezier curves \( P^I_n(u) \) and \( Q^I_n(u) \) are defined over the interval polygons \([p^-_1, p^+_1])_{i=0}^n \) and \([q^-_1, q^+_1])_{i=0}^n \), respectively. We may think of each curve as existing by itself, defined over the interval \( u \in [0,1] \) or some other interval. We may also think of the two interval curves as two interval segments of one composite interval Bezier curve, and we want to construct the interval Bezier curve \( Q^I_n(u) \) with control points \([q^-_1, q^+_1])_{i=0}^n \) so that \( Q^I_n(u) \) meets \( P^I_n(u) \) and matches the first \( n \) derivatives of \( P^I_n(u) \) at its end point.

This paper is organized as follows. Section II contains the basic results, whereas section III shows a numerical example and the final section offers conclusions.

II. THE BASIC RESULTS

Bezier curve provides a simple model of constructing a parametric curve. A parametric equation of Bezier curve can be defined by the linear combination of Bernstein polynomials and its interval control points. The interval control points can also be used for determining the shape of curve. Although, the curve does not pass through its interval control points except the first and the last interval endpoints, it passes closely to its interval control points. The interval Bezier curve is an interval polynomial of degree \( n \), which makes it slow to compute for large values of \( n \). It is therefore preferable to connect several Bezier segments, each defined by a few points, typically four to six, into one smooth interval curve. The conditions for smooth connection of two or more such interval segments can be derived as follows:

Let \([p^-_1, p^+_1])_{i=0}^n \) be a given set of interval control points which defines the interval Bezier curve:

\[ P^I_n(u) = \sum_{i=0}^n [p^-_i, p^+_i] B^I_k(u), \quad 0 \leq u \leq 1 \]  

of degree \( n \) where \( \{B^I_k(u)\}_{k=0}^n \) are Bernstein polynomials formed by:

\[ B^I_k(u) = \binom{l}{k} (1-u)^{(j-k)}u^k, \quad (k = 0, 1, \ldots, j) \]  

and

\[ \binom{l}{k} = \frac{j!}{k!(j-k)!} \]

is a binomial coefficient. Similarly, let \([q^-_1, q^+_1])_{i=0}^n \) be a given set of interval control points which defines another interval Bezier curve:

\[ Q^I_n(u) = \sum_{i=0}^n [q^-_i, q^+_i] B^I_k(u), \quad 0 \leq u \leq 1 \]  

For determining the shape of parametric derivatives are equal at that point creates a condition of continuity called \( n^{th} - order \) parametric continuity. Constructing curves from several segments can only succeed if the conditions for smooth connection \( (n^{th} - order \) parametric continuity) are satisfied. If the two interval Bezier curves defined in equations (1) and (2) to be joined smoothly at a common interval point \([p^+_n, p^-_n] \) = \([q^-_0, q^+_0] \), the following conditions have to be satisfied:

\[ \begin{align*}
P^I_n(0) &= Q^I_n(0) \\
(P^I_n)'(0) &= (Q^I_n)'(0) \\
& \vdots \\
(P^I_n)^{(n)}(0) &= (Q^I_n)^{(n)}(0)
\end{align*} \]

where, \( (D^I_n)^{(r)}(u) \) is the \( r^{th} \) derivative of \( D^I_n(u) \).

The four fixed Kharitonov's polynomials (four fixed Bezier curves) [17] associated with the first interval Bezier curve \( P^I_n(u) \) are:

\[ P^I_n(u) = p^-_0 + p^+_1u + p^-_2u^2 + p^+_3u^3 + p^-_4u^4 + p^+_5u^5 + \cdots \]

\[ P^I_n(u) = \alpha^I_{0,n} + \alpha^I_{1,n}u + \alpha^I_{2,n}u^2 + \cdots + \alpha^I_{n,n}u^n \]

\[ P^I_n(u) = p^-_0 + p^+_1u + p^-_2u^2 + p^+_3u^3 + p^-_4u^4 + p^+_5u^5 + \cdots \]

\[ P^I_n(u) = p^-_0 + p^+_1u + p^-_2u^2 + p^+_3u^3 + p^-_4u^4 + p^+_5u^5 + \cdots \]

\[ P^I_n(u) = p^-_0 + p^+_1u + p^-_2u^2 + p^+_3u^3 + p^-_4u^4 + p^+_5u^5 + \cdots \]

\[ P^I_n(u) = p^-_0 + p^+_1u + p^-_2u^2 + p^+_3u^3 + p^-_4u^4 + p^+_5u^5 + \cdots \]

\[ P^I_n(u) = p^-_0 + p^+_1u + p^-_2u^2 + p^+_3u^3 + p^-_4u^4 + p^+_5u^5 + \cdots \]

where, \( \{B^I_k(u)\}_{k=0}^n \) are Bernstein polynomials as in equation (2).

When two or more curves are joined end-to-end forming a string of curves, called a composite curve. Joining two or more curve segments at a common point so that their first \( n \) parametric derivatives are equal at that point creates a condition of continuity called \( n^{th} - order \) parametric continuity. Constructing curves from several segments can only succeed if the conditions for smooth connection \( (n^{th} - order \) parametric continuity) are satisfied. If the two interval Bezier curves defined in equations (1) and (2) to be joined smoothly at a common interval point \([p^+_n, p^-_n] \) = \([q^-_0, q^+_0] \), the following conditions have to be satisfied:

\[ \begin{align*}
P^I_n(0) &= Q^I_n(0) \\
(P^I_n)'(0) &= (Q^I_n)'(0) \\
& \vdots \\
(P^I_n)^{(n)}(0) &= (Q^I_n)^{(n)}(0)
\end{align*} \]

The four fixed Kharitonov's polynomials (four fixed Bezier curves) [17] associated with the first interval Bezier curve \( P^I_n(u) \) can be written as follows:

\[ P^I_n(u) = \sum_{i=0}^n a^I_{i,n}B^I_k(u) \]

for all \( u \in [0,1] \) and \( (j = 1,2,3,4) \)

The four fixed Kharitonov’s polynomials (four fixed Bezier curves) [17] associated with the second interval Bezier curve \( Q^I_n(u) \) are:

\[ Q^I_n(u) = q^-_0 + q^+_1u + q^-_2u^2 + q^+_3u^3 + q^-_4u^4 + q^+_5u^5 + \cdots \]

\[ Q^I_n(u) = \beta^I_{0,n} + \beta^I_{1,n}u + \beta^I_{2,n}u^2 + \cdots + \beta^I_{n,n}u^n \]

\[ Q^I_n(u) = q^-_0 + q^+_1u + q^-_2u^2 + q^+_3u^3 + q^-_4u^4 + q^+_5u^5 + \cdots \]

\[ Q^I_n(u) = q^-_0 + q^+_1u + q^-_2u^2 + q^+_3u^3 + q^-_4u^4 + q^+_5u^5 + \cdots \]

\[ Q^I_n(u) = q^-_0 + q^+_1u + q^-_2u^2 + q^+_3u^3 + q^-_4u^4 + q^+_5u^5 + \cdots \]

\[ Q^I_n(u) = q^-_0 + q^+_1u + q^-_2u^2 + q^+_3u^3 + q^-_4u^4 + q^+_5u^5 + \cdots \]

\[ Q^I_n(u) = q^-_0 + q^+_1u + q^-_2u^2 + q^+_3u^3 + q^-_4u^4 + q^+_5u^5 + \cdots \]

\[ Q^I_n(u) = q^-_0 + q^+_1u + q^-_2u^2 + q^+_3u^3 + q^-_4u^4 + q^+_5u^5 + \cdots \]

\[ Q^I_n(u) = q^-_0 + q^+_1u + q^-_2u^2 + q^+_3u^3 + q^-_4u^4 + q^+_5u^5 + \cdots \]
\[ Q_n^j(u) = q_0^j + q_1^j u + q_2^j u^2 + q_3^j u^3 + q_4^j u^4 + q_5^j u^5 + \ldots \]
\[ \equiv \beta_{0,n}^{j} + \beta_{1,n}^{j} u + \beta_{2,n}^{j} u^2 + \ldots + \beta_{n,n}^{j} u^n \]
\[ (7) \]

Similarly, the four fixed Kharitonov's polynomials (four fixed Bezier curves) associated with the second interval Bezier curve \( Q_n^j(u) \) can be written as follows:
\[ Q_n^j(u) = \sum_{i=0}^{n} \beta_{i,n}^{j} B_i^n(u) \]
\[ \text{for all } u \in [0,1] \text{ and } (j = 1,2,3,4) \]
\[ (8) \]

Now, joining two or more interval Bezier curve segments at a common point so that their first \( n \) parametric derivatives are equal at that point create conditions of continuity called \( n^{th-order} \) parametric continuity is equivalent to the conditions for smooth connections of the four fixed Kharitonov's polynomials \( P_j^i(u) \) (four fixed Bezier curves) associated with the first interval Bezier curve equation (6) with the corresponding four fixed Kharitonov's polynomials \( Q_n^j(u) \) (four fixed Bezier curves) associated with the second interval Bezier curve equation (8) at common points \( \alpha_{a,n}^{j} = \beta_{0,n}^{j} \) for \( j = 1,2,3,4 \). Thus, the conditions become:
\[ \begin{bmatrix}
P_1^1(1) & = & Q_1^0(0) \\
P_1^1(1) & = & Q_1^0(0) \\
P_1^0(1) & = & Q_1^0(0) \\
P_1^0(1) & = & Q_1^0(0) \\
\end{bmatrix}, \text{ for } j = 1,2,3,4 \]
\[ (9) \]

where, \( (P_n^j)^{(r)}(u) \) and \( (Q_n^j)^{(r)}(u) \) are the \( r^{th} \) derivatives of \( P_n^j(u) \) and \( Q_n^j(u) \), respectively. The \( r^{th} \) derivatives of \( (P_n^j)^{(r)}(u) \) and \( (Q_n^j)^{(r)}(u) \) for \( j = 1,2,3,4 \) can be calculated as follows:
\[ (P_n^j)^{(r)}(u) = \sum_{i=0}^{n-r} (a_{i,n}^{j})^{(r)} B_i^n(u) \]
\[ (10) \]
\[ (Q_n^j)^{(r)}(u) = \sum_{i=0}^{n-r} (\beta_{i,n}^{j})^{(r)} B_i^n(u) \]
\[ (11) \]

where,
\[ (a_{i,n}^{j})^{(r)} = n(n-1) \cdots (n-r+1) \sum_{k=0}^{r} (-1)^{r-k} \binom{r}{k} \alpha_{i+k,n}^{j} \]
\[ (12) \]
\[ (\beta_{i,n}^{j})^{(r)} = n(n-1) \cdots (n-r+1) \sum_{k=0}^{r} (-1)^{r-k} \binom{r}{k} \beta_{i+k,n}^{j} \]
\[ (13) \]

Using equation (14) with smoothness conditions in equation (9), we have:
\[ \sum_{k=0}^{r} (-1)^{r-k} \binom{r}{k} \alpha_{n-r+k,n}^{j} = \sum_{k=0}^{r} (-1)^{r-k} \binom{r}{k} \beta_{b,n}^{j} \]
\[ r = 0,1,\ldots,n \]
\[ (15) \]

which is a set of \( j(n+1) \) linear equations. In particular the \( l^{th} \) equation is:
\[ (-1)^l \binom{l}{0} \alpha_{n-l,n}^{j} + \cdots + (-1)^l \binom{l}{l-1} \alpha_{n-1,n}^{j} + (-1)^l \binom{l}{l} \alpha_{n,n}^{j} = \]
\[ (-1)^l \binom{l}{0} \beta_{n,n}^{j} + (-1)^{l-1} \binom{l}{l-1} \beta_{n-1,n}^{j} + \cdots + (-1)^l \binom{l}{l} \beta_{n,n}^{j} \]
\[ j = 1,2,3,4 \]
\[ (16) \]

If we look at equation \( l \) we realize that the \( (i+1) \) coefficients in each equation are simply the binomial coefficients in the expansion of \((-x+y)^l\). Each equation \( l \) is a linear combination of \( a_{i,n}^{j}, a_{i+1,n}^{j}, \ldots, a_{n,n}^{j} \) on the left hand side and a linear combination of \( \beta_{0,n}^{j}, \beta_{1,n}^{j}, \ldots, \beta_{n,n}^{j} \) on the right hand side, then the first \( (n-1) \) coefficients are zero and the rest are the binomial coefficients in the expansion of \((-x+y)^l\). Same remarks apply for the right hand side. The above set of linear equations can be written in matrix form as follows:
\[ A. \alpha^{j} = B. \beta^{j}, \text{ for } j = 1,2,3,4 \]
\[ (17) \]

where,
\[ A = \begin{bmatrix}
0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & (-1)^{l}\binom{l}{l} \\
0 & 0 & 0 & \cdots & (-1)^{l-1}\binom{l}{l-1} & (-1)^{l-1}\binom{l}{l-1} \\
0 & 0 & 0 & \cdots & (-1)^{l-2}\binom{l}{l-2} & (-1)^{l-2}\binom{l}{l-2} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & \cdots & (-1)^{l-1}\binom{l}{l-1} & \cdots & (-1)^{l-1}\binom{l}{l-1} & (-1)^{l-1}\binom{l}{l-1} \\
\vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\
0 & \cdots & \cdots & \cdots & \cdots & (-1)^{l}\binom{l}{l} \\
0 & \cdots & \cdots & \cdots & \cdots & (-1)^{l}\binom{l}{l} \\
0 & \cdots & \cdots & \cdots & \cdots & (-1)^{l}\binom{l}{l} \\
0 & \cdots & \cdots & \cdots & \cdots & (-1)^{l}\binom{l}{l} \\
\vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\
\end{bmatrix} \]

and
\[ B = \begin{bmatrix}
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\end{bmatrix} \]
As explained in section II, the four fixed Kharitonov’s polynomials (four fixed Bezier curves) associated with \( P_j(u) \) and \( Q_j(u) \) are obtained, and the matrices \( A, B \) and \( B^{-1} \) are found as follows:

\[
A = \begin{bmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & -1 & 1 \\
0 & 1 & -2 & 1 \\
-1 & 3 & -3 & 1
\end{bmatrix}, \quad B = \begin{bmatrix}
1 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 \\
1 & 2 & 1 & 0 \\
1 & 3 & 3 & 1
\end{bmatrix}
\]

\[
B^{-1} = \begin{bmatrix}
1 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 \\
1 & 2 & 1 & 0 \\
1 & 3 & 3 & 1
\end{bmatrix}
\]

The interval control points of the interval Bezier curves \( Q_j(u) \) for \( 0 \leq u \leq 1 \), can be obtained as:

\[
[q_\ell^- , q_\ell^+] = \left( [28.0000,33.0000] \times [18.0000,20.0000] \right) \\
[q_\ell^+ , q_\ell^- ] = \left( [30.0000,44.0000] \times [13.0000,19.0000] \right) \\
[q_{\ell+} , q_{\ell-} ] = \left( [32.0000,64.0000] \times [14.0000,27.0000] \right) \\
[q_{\ell+} , q_{\ell-} ] = \left( [26.0000,80.0000] \times [44.0000,68.0000] \right)
\]

IV. CONCLUSIONS

Bezier curves may be joined to form a composite curve. This allows us to create a more complex curve without having to raise the degree of an equivalent single curve. The geometric relationship of the control points adjacent to the joints determines the continuity conditions at those joints. The advantage of higher-order Bezier curves is that they provide correspondingly higher orders of continuity between segments of composite curves. However, we must consider how higher-degree polynomial functions affect the computation of geometric properties and relationships. Conditions for smooth connection of interval Bezier curve segments are presented in this paper. The conditions for joining two or more interval Bezier curve segments at a common point are converted into the conditions for smooth connections of the four fixed Kharitonov’s polynomials (four fixed Bezier curves) associated with the first interval Bezier curve with the corresponding four fixed Kharitonov’s polynomials (four fixed Bezier curves) associated with the second interval Bezier curve at common points \( \beta_{0,n} = \beta_{n,n} \) for \( j = 1,2,3,4 \). Finally, the interval control points of the second interval Bezier curve are obtained from conditions for smooth connections of the four fixed Kharitonov’s polynomials (four fixed Bezier curves) associated with the second interval Bezier curve. Two or more interval Bezier curve segments of like degree can be easily joined together to form what is termed a spline and this is important for animation applications where a changing viewpoint (a moving camera, for instance) is required without any abrupt changes in velocity and acceleration.
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