Natural Frequency Analysis of Coupled Liquid-Structure-Gas System in a Rigid Cylindrical Cavity

1, a M. H. Zainulabidin and 2, b D. G. Gorman
1 Faculty of Mechanical & Manufacturing Engineering, Universiti Tun Hussein Onn Malaysia, 86400 Parit Raja, Batu Pahat, Johor, Malaysia
2 Department of Mechanical Engineering, University of Strathclyde, James Weir Building, Montrose Street, G1 1XJ, Glasgow, UK

Abstract— This paper describes the natural frequency analysis of strongly coupled vibration of a closed ends rigid cylindrical cavity containing liquid and gas which separated by a thin circular plate at their interface. A theoretical analysis method to compute the natural frequencies of the liquid-structure-gas coupled system were developed with assumptions is made with respect to the boundaries between the liquid, structure and gas. The theoretical natural frequencies of the complete coupled system were found to be in good agreement with the corresponding values obtained experimentally and from a commercial Finite Element Analysis (FEA) code.

Index Term— Natural frequency, fluid-structure interaction, vibroacoustic.

1. INTRODUCTION

The growing deployment of thin walled structure in a wide range of areas has challenged researchers to understand more of the science of fluid-structure vibration interaction. The benefits of modern thin walled structures are they are light in weight, easy to shape, have high strength to weight ratios and have high corrosion and erosion resistant properties. However due to the slenderness of these types of structures, their vibration behaviour is often influenced by their surroundings to such an extent that their vibration characteristics can no longer can be analysed alone. For this reason the study of fluid-structure vibration interaction is very important.

Several cases of free vibration of a thin structure in contact with fluid have been studied recently. Jeong and Kim [1] studied the hydroelastic vibration of a circular plate submerged in a bounded compressible fluid. An analytical method to compute the natural frequencies of a circular plate submerged in a fluid had been developed. The effect of the plate position on the natural frequencies was observed. It was found that the natural frequencies drastically decrease when the plate approaches the rigid cylindrical container’s bottom or top surface. Gorman et al. [2] analysed the vibration of a circular disc backed by a cylindrical cavity containing gas. A theoretical method to calculate the natural frequencies of a vibrating circular plate in interaction with a closed volume of compressible fluid had been developed. Gorman et al. [3] extended the analysis where the modal parameter such as modal energy of the interacting fluid-structure system is extracted. The vibration of a thin circular plate influenced by liquid-gas interaction in a cylindrical cavity was studied by Gorman et al. [4]. A theoretical method to calculate the natural frequencies of a system consists a cylindrical gas cavity axially bounded at the bottom end by a liquid and a thin elastics circular plate at the top end. Gorman et al. [5] described the characteristics of strongly coupled membrane-fluid column interaction by simplified mass-spring system model. The mass-spring model compared fairly well with the full coupled model. The vibration analysis of a circular membrane backed by cylindrical cavity using multimodal approach was studied by Rajalingam et al. [6]. The cavity-backed membrane was modeled as a dynamical system composed of two subsystems, and their modal receptance characteristics were used to study the system vibration. Recently, Zainulabidin and Gorman [7] studied the axial vibration analysis of a closed ends rigid cylindrical container containing liquid and gas which separated by a thin circular plate at their interface. The natural frequencies obtained experimentally were compared favorably with those of commercial FEA software, ANSYS. It was found that strong coupling predominantly occurs between liquid and structure. In weak coupling conditions, the modes are predominantly gas mode.

The aim of this paper is to present the theoretical analysis of a coupled liquid-structure-gas vibration system in a rigid cylindrical cavity and then the computed theoretical values will be compared with the values obtained experimentally and by FEA. The studied system is composed of a cylindrical thick-walled cavity containing two compressible inviscid fluid inside; liquid and gas. The liquid at the bottom is separated from the gas at the top by a thin circular plate which is clamped at its circumferential edge. The schematic diagram of the studied system is presented in Fig. 1.
2. THEORETICAL ANALYSIS

The equation of motion describing the free small lateral vibration, \( w = w(r, \theta, t) \), of a thin circular plate in interaction with the acoustic cavities, as shown in Fig. 1, is

\[
\nabla^4 \bar{w} = -\frac{\rho_a h a^4}{D_o} \frac{\partial^2 \bar{w}}{\partial t^2} + \frac{p_1 a^3}{D_o} \bigg|_{\tau_1=1} - \frac{p_2 a^3}{D_o} \bigg|_{\tau_2=1}
\]

Eqn. (1)

where

\[
\nabla^4 = \left( \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial}{\partial \theta^2} \right)^2
\]

\[
\bar{w} = \frac{w}{a}, \quad \bar{r} = \frac{r}{a}, \quad \bar{x}_1 = \frac{x_1}{l_1}, \quad \bar{x}_2 = \frac{x_2}{l_2} \quad \text{and} \quad D_o = \frac{E h^3}{12(1-\nu^2)};
\]

\( E \) is Young’s modulus, \( \nu \) is Poisson ratio and \( \rho_a \) is the plate mass density; \( a \) and \( h \) are the radius and thickness of the plate respectively; \( l_1 \) and \( l_2 \) are the length of the cylindrical liquid cavity and gas cavity respectively and \( p_1 \) and \( p_2 \) are the water and gas pressures at the surface of the plate respectively.

Now writing

\[
\bar{w} = \sum_{m=0}^{\infty} \bar{W}_m,
\]

Eqn. (2)

where

\[
\bar{W}_m = \bar{W}_m(\bar{r}) \cos m\theta e^{j\omega t}
\]

and

\[
W_m(\bar{r}) = \sum_{s=1}^{\infty} \chi_m \Psi_{ms}(\bar{r}),
\]

Eqn. (3)
where \( \psi_{ms}(\vec{r}) \) is the natural mode shape of the disc in vacuo and \( \chi_{ms} \) is a constant for that mode, generally referred to as the mode shape coefficient for the mode consisting of \( m \) nodal diameters and \( s \) nodal circles. In this particular case, for a disc clamped at the periphery, the mode shapes, \( \psi_{ms}(\vec{r}) \), are according to Leissa [9]:

\[
\psi_{ms}(\vec{r}) = -\frac{I_m(\xi_{ms})}{J_m(\xi_{ms})} J_m(\xi_{ms} r) + I_m(\xi_{ms} r),
\]

where \( \xi_{ms} \) are roots for value of \( s = 1, 2, 3 \) etc. computed from equation:

\[
0 = \xi_{ms} J_{m+1}(\xi_{ms}) - \xi_{ms} \frac{J_{m+1}(\xi_{ms})}{J_m(\xi_{ms})} I_m(\xi_{ms} r),
\]

where \( I_m \) and \( J_m \) are Bessel functions.

For particular values of \( m \) and \( s \), the natural frequency of free undamped vibration, \( \omega_{ms} \), is then:

\[
\omega_{ms} = \xi_{ms}^2 \sqrt{\frac{D_o}{\rho_s h a^4}}.
\]

Now for a particular mode of vibration for the disc in vacuo:

\[
\nabla^4 [\psi_{ms}(\vec{r}) \cos m\theta] = \frac{\omega_{ms}^2}{D_o} \frac{\rho_s h a^4}{\rho_D h a^4} \psi_{ms}(\vec{r}) \cos m\theta.
\]

Therefore combination of Eqn. (1), (3) and (7) gives:

\[
\sum_{m=0}^{\infty} \sum_{s=1}^{\infty} \left[ (\omega_{ms}^2 - \omega^2) \chi_{ms} \psi_{ms}(\vec{r}) \right] \cos m\theta e^{j\omega t} = \frac{p_1}{\rho_D h a^3} - \frac{p_2}{\rho_D h a^3}.
\]

We shall now establish the form of the acoustic pressure, \( p_i \) \((i = 1 \text{ (liquid)} \text{ and } i = 2 \text{ (gas)})\) acting on the disc by reference to the acoustic cavity. Consider the acoustic cavity shown in Figure 1, whose velocity potential, \( \phi = \phi(x, r, \theta, t) \) is described by

\[
\frac{\partial^2 \phi_i}{\partial \vec{r}^2} + \frac{1}{r} \frac{\partial \phi_i}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \phi_i}{\partial \theta^2} + \left( \frac{a}{l_i} \right)^2 \frac{\partial^2 \phi_i}{\partial x_i^2} = \left( \frac{a}{c_i} \right)^2 \frac{\partial^2 \phi_i}{\partial t^2},
\]

where \( \phi_i = \frac{\phi_i}{ac_i}, \bar{x}_i = \frac{x_i}{l_i}, \bar{r} = \frac{r}{a} \) and \( c_i \) is speed of sound.

Now writing

\[
\bar{\phi}_i = \sum_{m=0}^{\infty} \phi_{im}^2,
\]

where \( \phi_{im} = H_{im}(\bar{x}_i)Q_{im}(\bar{r}) \cos m\theta e^{j\omega t} \).

\[
\]

\[
\]
Substituting Eqn. (10) into (9) gives (for a set value of \( m \))

\[
\left( \frac{a}{l_i} \right)^2 \frac{H''_{im}}{H_{im}} = - \left[ \frac{Q''_{im}}{Q_{im}} + \frac{1}{\bar{r}} \frac{Q'_{im}}{Q_{im}} - \frac{m^2}{\bar{r}^2} + \lambda_i^2 \right] = \pm k_i^2,
\]

Eqn. (11)

where \( \lambda_i = \frac{\alpha a}{c_i} \) and \( k_i \) is a constant.

For the right hand side of Eqn. (11) equal to \(-k_i^2\) we have

\[
Q_{im}(\bar{r}) = B_{im} J_m(\alpha \bar{r}) + \tilde{B}_{im} Y_m(\alpha \bar{r}),
\]

where \( \alpha = \sqrt{\lambda_i^2 - k_i^2} \) or \( k_i = \sqrt{\lambda_i^2 - \alpha^2} \), \( \tilde{B}_i = 0 \) since \( Q_{im}(\bar{r}) \) must be finite when \( \bar{r} \to 0 \).

At \( \bar{r} = 1 \) for each value of \( m \)

\[
\frac{\partial \phi_{im}}{\partial \bar{r}} \equiv \frac{dQ_{im}}{d\bar{r}} = 0.
\]

Eqn. (12)

Therefore for a set value of \( m \), the Eqn. (12) has roots \( \alpha_{mq} \) \( (q = 1, 2, 3 \text{ etc.}) \), which satisfy the equation \( J'_m(\alpha) = 0 \).

Similarly, since \( \frac{dH_{im}}{d\bar{x}_i} \bigg|_{\bar{x}_i=0} = 0 \)

\[
H_{im} = C \cos(\gamma_{mq}(\bar{x}_i)),
\]

where \( \gamma_{mq}(\alpha) = \frac{l_i}{a} \sqrt{\lambda_i^2 - \alpha_{mq}^2} = \frac{l_i}{a} k_{mq}(\alpha) \).

Therefore Eqn. (10), for a set value of \( m \) becomes:

\[
\bar{\phi}_{im} = \sum_{q=1}^{\infty} B_{mq} \cos(\gamma_{mq}(\bar{x}_i)) J_m(\alpha_{mq} \bar{r}) \cos m\theta e^{j\omega t}.
\]

Eqn. (13)

At \( \bar{x}_i = 1 \), the axial component of the velocity of the liquid must be equal to the lateral velocity of the plate, i.e.,

\[
\frac{c_i}{l_i} \frac{\partial \bar{\phi}_{im}}{\partial \bar{x}_i} \bigg|_{\bar{x}_i=1} = \frac{\partial \bar{w}_m}{\partial t}
\]

Therefore from Eqn. (2), (3) and (13) for a set value of \( m \) we have

\[
-\frac{c_i}{l_i} \sum_{q=1}^{\infty} [B_{mq} \gamma_{mq}(\alpha) \sin(\gamma_{mq}(\alpha)) J_m(\alpha_{mq} \bar{r})] = j \omega \sum_{s=1}^{\infty} \chi_{ms} \psi_{ms}(\bar{r}).
\]

Eqn. (14)
Now using the orthogonal properties of the eigenfunction, \( \bar{r} J_m (\alpha_{mq} \bar{r}) \), when we multiplying both sides of Eqn. (14) by \( \bar{r} J_m (\alpha_{mq} \bar{r}) \) and integrating between \( 0 \leq \bar{r} \leq 1 \) gives

\[
B_{1mq} = -2 j \omega l_1 \sum_{s=1}^{\infty} \chi_{ms} K_{msq} c_1 \gamma_{1mq}^{(ao)} \sin(\gamma_{1mq}^{(ao)}) \left( 1 - \frac{m^2}{\alpha_{mq}^2} \right) J_m^2 (\alpha_{mq}), \tag{15}
\]

where \( K_{msq} = \int_0^1 \bar{r} \psi_{ms} (\bar{r}) J_m (\alpha_{mq} \bar{r}) d\bar{r}. \tag{16} \)

Similarly, at \( \bar{x}_2 = 1 \), the axial component of the velocity of the gas must be equal to the lateral velocity of the plate, i.e.,

\[
- \frac{c_2}{l_2} \frac{\partial \bar{\phi}_{2m}}{\partial \bar{x}_2} \bigg|_{\bar{x}_2=1} = \frac{\partial \bar{W}_m}{\partial \bar{t}}.
\]

From equations (2), (3) and (13) for a set value of \( m \) we have

\[
\frac{c_2}{l_2} \sum_{q=1}^{\infty} \left[ B_{2mq} \gamma_{2mq}^{(ao)} \sin(\gamma_{2mq}^{(ao)}) J_m (\alpha_{mq} \bar{r}) \right] = j \omega \sum_{s=1}^{\infty} \chi_{ms} \psi_{ms} (\bar{r}). \tag{17}
\]

Multiplying both sides of Eqn. (14) by \( \bar{r} J_m (\alpha_{mq} \bar{r}) \) and integrating between \( 0 \leq \bar{r} \leq 1 \) gives

\[
B_{2mq} = 2 j \omega l_2 \sum_{s=1}^{\infty} \chi_{ms} K_{msq} c_2 \gamma_{2mq}^{(ao)} \sin(\gamma_{2mq}^{(ao)}) \left( 1 - \frac{m^2}{\alpha_{mq}^2} \right) J_m^2 (\alpha_{mq}), \tag{18}
\]

where \( K_{msq} = \int_0^1 \bar{r} \psi_{ms} (\bar{r}) J_m (\alpha_{mq} \bar{r}) d\bar{r}. \)

Now the pressure, \( p_i \) at the surface of the plate is given by Fahy [10]:

\[
p_i = - \rho_i ac \frac{\partial \bar{\phi}}{\partial \bar{t}} \bigg|_{\bar{x}_2=1},
\]

where \( \rho_i \) is the fluid density i.e. \( \rho_i \) is the liquid density and \( \rho_2 \) is the gas density.

Therefore combining Eqn. (13) and (15) we have:
\[ p_1 = -2\omega^2 a_1 \rho_1 \sum_{s=1}^{\infty} \sum_{q=1}^{\infty} \chi_{ms} K_{mq} J_m(\alpha_{mq}) \frac{\chi_{ms} K_{mq} J_m(\alpha_{mq})}{\left( \gamma_{1mq}^{(ao)} \tan \gamma_{1mq}^{(ao)} \left(1 - \frac{m^2}{\alpha_{mq}^2} \right) J_m^2(\alpha_{mq}) \right)} \cos m\theta e^{i\omega t}, \quad \text{Eqn. (19)} \]

and combining Eqn. (13) and (18) we have:

\[ p_2 = 2\omega^2 a_2 \rho_2 \sum_{s=1}^{\infty} \sum_{q=1}^{\infty} \chi_{ms} K_{mq} J_m(\alpha_{mq}) \frac{\chi_{ms} K_{mq} J_m(\alpha_{mq})}{\left( \gamma_{2mq}^{(ao)} \tan \gamma_{2mq}^{(ao)} \left(1 - \frac{m^2}{\alpha_{mq}^2} \right) J_m^2(\alpha_{mq}) \right)} \cos m\theta e^{i\omega t}, \quad \text{Eqn. (20)} \]

Substituting Eqn. (19) and (20) into equation (8) gives

\[ \sum_{s=1}^{\infty} \left( \omega_{ms}^2 - \omega^2 \right) \chi_{ms} \psi_{ms}(\bar{\tau}) = -2\omega^2 \frac{\rho_1 l_1}{h} \sum_{s=1}^{\infty} \sum_{q=1}^{\infty} \chi_{ms} K_{mq} J_m(\alpha_{mq}) \left( \gamma_{1mq}^{(ao)} \tan \gamma_{1mq}^{(ao)} \left(1 - \frac{m^2}{\alpha_{mq}^2} \right) J_m^2(\alpha_{mq}) \right), \quad \text{Eqn. (21)} \]

Multiplying both sides by \( \bar{\tau} J_m(\alpha_{mq}) \) and integrating between \( 0 \leq \bar{\tau} \leq 1 \) we get

\[ \sum_{s=1}^{\infty} \chi_{ms} K_{mq} \left( \omega_{ms}^2 - \omega^2 \left[ 1 - \frac{\Omega_1}{\gamma_{1mq}^{(ao)} \tan \gamma_{1mq}^{(ao)}} - \frac{\Omega_2}{\gamma_{2mq}^{(ao)} \tan \gamma_{2mq}^{(ao)}} \right] \right) = 0, \quad \text{Eqn. (22)} \]

where \( \Omega_1 = \frac{\rho_1 l_1}{\rho_D h} = \frac{\text{mass of liquid}}{\text{mass of plate}} \) and \( \Omega_2 = \frac{\rho_2 l_2}{\rho_D h} = \frac{\text{mass of liquid}}{\text{mass of plate}} \).

We can introduce a quantity \( \xi \) instead of \( \omega \) by the relation

\[ \omega^2 = \xi^4 \frac{D}{\rho_D ha^4}. \quad \text{Eqn. (23)} \]

Hence Eqn. (22) can be re-written as

\[ \sum_{s=1}^{\infty} \chi_{ms} K_{mq} \left( \xi_{ms}^4 - \xi^4 \left[ 1 - \frac{\Omega_1}{\gamma_{1mq}^{(ao)} \tan \gamma_{1mq}^{(ao)}} - \frac{\Omega_2}{\gamma_{2mq}^{(ao)} \tan \gamma_{2mq}^{(ao)}} \right] \right) = 0, \quad \text{Eqn. (24)} \]

\[ \begin{array}{c}
\frac{153902.7474-IJME-IJENS © April 2015 IJENS}
\end{array} \]
where \( \gamma_{1mq}^{(\xi)} = \frac{l_i}{a} \sqrt{\frac{D_o}{\rho_D h a^2 c_i^2} - \alpha_{mq}^2} \)

Eqn. (24) can be represented in matrix form as

\[
\begin{bmatrix}
  a_{11}(\xi) & a_{12}(\xi) & \cdots & a_{1N}(\xi) \\
a_{21}(\xi) & a_{22}(\xi) & \cdots & a_{2N}(\xi) \\
  \vdots & \vdots & \ddots & \vdots \\
  \vdots & \vdots & \cdots & \vdots \\
a_{N1}(\xi) & a_{N2}(\xi) & \cdots & a_{NN}(\xi)
\end{bmatrix}
\begin{bmatrix}
  \chi_{m1} \\
  \chi_{m2} \\
  \vdots \\
  \vdots \\
  \chi_{mN}
\end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ \vdots \\ 0 \end{bmatrix}
\tag{25}
\]

where

\[
a_{q1}(\xi) = K_{mq}\left\{\left(\frac{\xi}{\gamma_{mq}}\right)^2 - \left(\frac{\xi}{\gamma_{1mq}}\right)^2\right\}^{1 - \Omega_1} - \frac{\Omega_2}{\gamma_{2mq}^2 \tan \gamma_{2mq}^2}
\tag{26}
\]

Hence values of \( \xi^2 \) can be obtained (iterated upon) which renders the determinant of matrix Eqn. (25) equal to zero. Consequently for each of these values of \( \xi^2 \) can then obtain the corresponding values of mode shape coefficients \( \chi_{m1}, \chi_{m2}, \ldots, \chi_{mN} \) normalized to \( \chi_{m1} \). The determinant is obtained by performing the LU decomposition. The value of the determinant is the product of the diagonal terms. This LU decomposition method is adapted from Press et. al. [11]. The values of \( \xi^2 \) which render the determinant zero are substituted back into Eqn. (25) to obtain the corresponding values of the mode shape coefficients, \( \chi_{m1} \). The above theory has been implemented by a program written in FORTRAN code.

3. RESULTS AND DISCUSSION

Nine liquid-disc-coupled systems with different values of \( l_i/l_2 \) were considered. The value of \( l_i/l_2 \) was varied from 0.125 to 8. The disc thickness was 0.64 mm and the disc radius was 40 mm and assumed clamped all along the periphery. The density of liquid, \( \rho_1 = 1000 \text{ kg/m}^3 \), speed of sound in liquid, \( c_1 = 1500 \text{ m/s} \), density of gas, \( \rho_2 = 1.2 \text{ kg/m}^3 \), speed of sound in gas, \( c_2 = 343 \text{ m/s} \), density of disc, \( \rho_D = 7800 \text{ kg/m}^3 \), Young’s modulus of disc, \( E = 2.1 \times 10^6 \text{ Pa} \). All these parameters are identical to the parameters used by Zainulabidin and Gorman [7].

Results of the analysis are presented in Table I. For verification purpose, the computed values are compared with the values obtained experimentally and by FEA done by Ref. [7]. The percentage of discrepancies between theoretical and experimental values, \( \kappa \) and the percentage of discrepancies between theoretical and FEA values, \( \tau \) are computed using Eqn. (27) and (28) respectively.

\[
\kappa = \frac{\xi_{\text{Exp}}^2 - \xi_{\text{Theo}}^2}{\xi_{\text{Theo}}^2} \times 100\%
\tag{27}
\]

\[
\tau = \frac{\xi_{\text{FEA}}^2 - \xi_{\text{Theo}}^2}{\xi_{\text{Theo}}^2} \times 100\%
\tag{28}
\]

The comparison of the first three natural frequency roots of liquid-disc-gas system between values obtained theoretically, experimentally and by FEA were presented. Referring to Table I, it can be seen that the natural frequency values of the liquid-disc-gas coupled system obtained experimentally compared favourably with the values obtained theoretically. The percentages of discrepancies between theoretical and experimental values are less than 6.7%. The percentages of discrepancies between theoretical and FEA values are very small which are less than 1%. Comparison of the theoretical, FEA and experimental natural frequency values are presented graphically in Figure 2(a)-(c).

There are three main reasons for errors between the theoretical and experimental values. The first one is due to the compliance of the actual boundary conditions as compared to the clamped boundary condition of the theoretical model. The constraints along the plate’s circumference in experimental
work are not uniform and the stiffness is not constant due to the bolts. The actual boundary conditions will no doubts lie somewhere between clamped and simply supported, albeit, much more towards the clamped conditions. The second source is the result of plate clamping by the cylinder wall itself. Plate clamping by the cylinder wall will generate either a compressive or tensile in-plane stress inside the plate due to the Poisson effect. The in-plane stress can change the natural frequency values of the plates. This effect is similar to the case of plate fastened with bolts around its circumference studied by Amabili et. al. [12]. The third source is the mass of the small miniature accelerometer concentrating at the disc centre. The effect of the accelerometer weight upon the thin plate boundary condition will always remain in the background when considering the results. The analysis of the accelerometer weight effect has been presented by Zainulabidin and Gorman [7, 8].

The less accurate agreement between theoretical and FEA values is mainly due to the important different in this two methods. In the theoretical analysis, it is implied that at the interface between the fluids and structure, only axial displacement is compatible. However, in FEA, the fluids-structure interfaces are coupled by complete compatibility in all directions.

Table I
Comparison of first three natural frequency roots of liquid-disc-gas system between values obtained theoretically, experimentally and by FEA

<table>
<thead>
<tr>
<th>( l_1/l_2 )</th>
<th>( \xi^2_{\text{Theo}} )</th>
<th>( \xi^2_{\text{Exp}} ) (Ref.[7])</th>
<th>( \kappa(%) )</th>
<th>( \xi^2_{\text{FEA}} ) (Ref. [7])</th>
<th>( \tau(%) )</th>
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Fig. 2(a)-(c). Comparison of the natural frequencies for the 1st, 2nd and 3rd mode respectively; — Theoretical, □ Experimental, Δ FEA values.

CONCLUSION
A theoretical analysis describing the free vibration of a liquid-disc-gas coupled system has been presented. The theoretical analysis method were developed with assumptions is made with respect to the boundaries between the liquid, structure and gas. The theoretical natural frequencies computed
theoretically have been compared favorably with the values obtained experimentally and by finite element analysis.

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REFERENCES