On The Use of the Scaled Boundary Finite Element Method for Dynamic Analysis of Plates

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Index Terms-- Scaled boundary finite element method, plate, vibration, frequency response function.

Abstract-- Predicting the dynamic behavior of complex structures is possible by the use of high accurate numerical models. Generally, structural systems are modeled using methods such as finite element method (FEM), finite difference method (FDM) or boundary element method (BEM). The scaled boundary finite element method (SBFEM) is a semi-analytical boundary method that does not require the knowledge of a fundamental-solution to make possible its formulation and it is based on FEM. Using in-plane and out of plane motion structural dynamic problems and discretizing the boundary with three nodes quadratic line finite element, in this work the derivation of the frequency response function (FRF) by SBFEM is presented. From the FRFs, the natural frequencies and mode shapes can be obtained by performing a theoretical modal analysis. Numerical studies considering various assumptions for the scaled boundary finite element method are presented and the results obtained are compared with FEM.

1. INTRODUCTION

The knowledge about a particular structure is contained in a theoretical model that can be constructed using a numerical method. A large number of papers written on the subject have used the finite element method (FEM) as the base for numerical model. The boundary element method (BEM) has been successfully applied to numerical solution of many engineering problems too, and for some classes of problems has been proved competitive with the FEM (Banerjee and Butterfield, [1]).

The scaled boundary finite element method (SBFEM) has been applied to many fields of engineering (Wolf & Song [2]). This method is based entirely on finite elements, but with the discretization only of the boundary. It combines the advantages of BEM and FEM. No fundamental solution is required, the spatial dimension is reduced by one, the radiation condition is satisfied exactly at infinity, and it yields a symmetric dynamic-stiffness matrix. Moreover, the boundary placed in the radial direction to the scaling center does not need to be discretized. Therefore a correct choice of the scaling center yields, for example, that cracks and the free surface of a foundation embedded in half-space can be modeled without discretization. In the following, a brief theoretical review of the scaled boundary finite element formulation for two-dimensional elastodynamics, according to Song and Wolf [3], will be presented, and a method to use it to model the dynamic behavior of plates under plane stress proposed and evaluated.

2. GEOMETRY TRANSFORMATION

The concept of scaled boundary finite element method can be explained by using a section of a 2D linear elastic bounded medium (Fig. 1). A scaling center O, located inside the domain, is chosen in a position from which the whole boundary must be reached. Only the boundary needs to be discretized with line finite elements. A radial coordinate \( \xi \), from the scaling center \( O \) to the boundary surface, and two local curvilinear coordinates \( \eta, \zeta \), in the circumferential directions of boundary surface are defined. The side-face \( A^e \), obtained by using straight lines to connect finite element surface to scaling center \( O \) and surface \( S^e \) compose a pyramid with volume \( V^e \), which is defined by the scaled boundary coordinates \( \xi, \eta \), with \( \xi = 0 \) in the scaling center and \( \xi = 1 \) on the boundary. This dimensionless radial coordinate \( \xi \) can be seen as a scaling factor or a characteristic length. The geometry transformation from Cartesian coordinates \( \hat{x}, \hat{y} \) to scaled boundary coordinates \( \xi, \eta \) is performed, which is equivalent to represent the surface in polar coordinates where the value of the radial coordinate at the boundary is constant. All triangles are assembled by connecting their side-faces, which corresponds to enforcing compatibility and equilibrium, resulting in the total medium with area \( A \) and the closed boundary \( S \).

Fig. 1. Scaled boundary coordinate system for a 2D domain.
Denoting points on the boundary with \( x, y \) the geometry in a 2D domain is described as
\[
\begin{align*}
\hat{x} (\xi, \eta) &= \xi N(\eta)x \\
\hat{y} (\xi, \eta) &= \xi N(\eta)y
\end{align*}
\] (1)
where \( N(\eta) \) is a shape function, and \( 0 \leq \xi \leq 1 \) from de scaling center to the boundary.

3. ELASTODYNAMIC EQUATIONS
The matrix differential equation in the frequency domain for in plane elastodynamics problems can be expressed in function of stresses \( \sigma \) and displacement amplitudes \( u \) as:
\[
\mathbf{L}^T \sigma + \omega^2 \rho \mathbf{u} = 0
\] (2)
where \( \rho \) is the mass density, and
\[
\mathbf{u} = \mathbf{u} (\hat{x}, \hat{y}) = \begin{bmatrix} u_x & u_y \end{bmatrix}^T
\] (3)
\[
\mathbf{\sigma} = \begin{bmatrix} \sigma_x & \sigma_y & \tau_{xy} \end{bmatrix}^T
\] (4)
\[
\mathbf{L} = \begin{bmatrix}
\frac{\partial}{\partial \xi} & 0 \\
0 & \frac{\partial}{\partial \eta} \\
\frac{\partial}{\partial \eta} & \frac{\partial}{\partial \xi}
\end{bmatrix}
\] (5)

Applying the scaled boundary transformation to the 2D domain geometry, the governing equations can be expressed as:
\[
\mathbf{E}^0 \xi^2 \mathbf{u}(\xi) + \left( \mathbf{E}^0 + \mathbf{E}^1 + (\mathbf{E}^1)^T \right) \xi \mathbf{u}(\xi) + \left( (\mathbf{E}^1)^T - \mathbf{E}^1 \right) \mathbf{u}(\xi) + \omega^2 \mathbf{M}^0 \xi \mathbf{u}(\xi) = 0
\] (6)
where
\[
\begin{align*}
\mathbf{E}^0 &= \int_{-1}^{1} \mathbf{B}^T \mathbf{D} \mathbf{B} \left| J \right| d\eta \\
\mathbf{E}^1 &= \int_{-1}^{1} \mathbf{B}^T \mathbf{D} \mathbf{B} \left| J \right| d\eta \\
\mathbf{E}^2 &= \int_{-1}^{1} \mathbf{B}^T \mathbf{D} \mathbf{B} \left| J \right| d\eta \\
\mathbf{M}^0 &= \int_{-1}^{1} \mathbf{N}^0(\eta) \mathbf{\rho} \mathbf{N}^0(\eta) \left| J \right| d\eta
\end{align*}
\] (7-10)
where \( \mathbf{D} \) is the elasticity matrix, and
\[
\begin{align*}
\mathbf{B}^1 &= \mathbf{b}^1 \mathbf{N}^0(\eta) \\
\mathbf{B}^2 &= \mathbf{b}^2 \mathbf{N}^0(\eta)
\end{align*}
\] (11-12)
\[
\mathbf{b}^1 = \frac{1}{\left| J \right|} \begin{bmatrix} y_{,\eta} & 0 \\
0 & -x_{,\eta} \\
-x_{,\eta} & y_{,\eta} \end{bmatrix} ; \quad \mathbf{b}^2 = \frac{1}{\left| J \right|} \begin{bmatrix} -y & 0 \\
0 & x \\
x & -y \end{bmatrix}
\] (13)
\[
\mathbf{J} = x_{,\eta} - y_{,\eta}; \quad \mathbf{N}^0(\eta) = \begin{bmatrix} N_1(\eta) & N_2(\eta) & \ldots \end{bmatrix}
\] (14)

4. DYNAMIC STIFFNESS MATRIX ON BOUNDARY
The dynamic stiffness matrix on a surface with constant \( \xi \) is defined as
\[
\mathbf{R}(\xi) = \mathbf{S}(\omega, \xi) \mathbf{u}(\xi)
\] (17)
where \( \mathbf{R}(\xi) \) are the amplitudes of nodal forces and \( \mathbf{S}(\omega) \) is the dynamic stiffness matrix.

For any surface \( \mathcal{S} \) with a constant \( \xi \), the virtual work is formulated as
\[
\mathbf{w}(\xi) \mathbf{R}(\xi) = \int_{\mathcal{S}} \mathbf{w}^T \mathbf{t}^\xi \ d\mathcal{S}^\xi
\] (18)
where \( \mathbf{t}^\xi \) are the amplitudes of the surface tractions, \( d\mathcal{S}^\xi \) the infinitesimal surface and \( \mathbf{w} \) is the weighting function.

For an arbitrary \( \mathbf{w}(\xi) \) and introducing \( \mathbf{E}^0, \mathbf{E}^1 \) and the displacements \( \mathbf{u}(\omega) \) in (18), results in
\[
\mathbf{R}(\xi) = \mathbf{E}^0 \xi^2 \mathbf{u}(\xi, \xi) + (\mathbf{E}^1)^T \xi \mathbf{u}(\xi)
\] (19)

Equating the right-hand sides of (17) and (19) yields
\[
\mathbf{S}(\omega, \xi) \mathbf{u}(\xi) = \mathbf{E}^0 \xi^2 \mathbf{u}(\xi, \xi) + (\mathbf{E}^1)^T \xi \mathbf{u}(\xi)
\] (20)

Differentiating (20) and adding with (6) results in
\[
\mathbf{S}(\omega, \xi) \mathbf{u}(\xi) + \left( \mathbf{S}(\omega, \xi) - \xi \mathbf{E}^1 \right) \mathbf{u}(\xi) = -\mathbf{E}^0 \mathbf{u}(\xi) + \omega^2 \mathbf{M}^0 \xi \mathbf{u}(\xi) = 0
\] (21)

Solving (21) for \( \mathbf{u}(\omega, \xi) \) and substituting the result in (21) leads to
\[
\left( \mathbf{S}(\omega, \xi) - \xi \mathbf{E}^1 \right) \left( \mathbf{E}^0 \right)^{-1} \left( \mathbf{S}(\omega, \xi) - \xi \left( \mathbf{E}^1 \right)^T \right) - \xi \mathbf{E}^0 + \xi \mathbf{S}(\omega, \xi) \mathbf{E}^1 + \omega^2 \xi \mathbf{M}^0 = 0
\] (22)

being
\[
\xi \mathbf{S}(\omega, \xi) \mathbf{E}^1 = \mathbf{S}(\omega, \xi) + \omega \mathbf{S}(\omega, \xi)
\] (23)

and substituting in (22) results in
\[
\left( \mathbf{S}(\omega, \xi) - \xi \mathbf{E}^1 \right) \left( \mathbf{E}^0 \right)^{-1} \left( \mathbf{S}(\omega, \xi) - \xi \left( \mathbf{E}^1 \right)^T \right) - \xi \mathbf{E}^0 - \mathbf{S}(\omega, \xi) + \omega \mathbf{S}(\omega, \xi) + \omega^2 \mathbf{M}^0 = 0
\] (24)

For the boundary \( (\xi = 1) \), the plate differential equation in terms of the dynamic stiffness matrix is given by
\[
\left( \mathbf{S}(\omega) - \mathbf{E}^1 \right) \left( \mathbf{E}^0 \right)^{-1} \left( \mathbf{S}(\omega) - \mathbf{E}^1 \right)^T - \mathbf{E}^2 + \mathbf{S}(\omega) + \omega \mathbf{M}^0 = 0
\] (25)

5. STATIC STIFFNESS AND MASS MATRICES ON BOUNDARY
The static stiffness matrix for the boundary is defined from (25) by setting \( \omega = 0 \):
\[
\left( \mathbf{K} - \mathbf{E}^1 \right) \left( \mathbf{E}^0 \right)^{-1} \left( \mathbf{K} - \mathbf{E}^1 \right)^T - \mathbf{E}^2 + \mathbf{K} = 0
\] (26)
where \( \mathbf{K} = \mathbf{S}(0) \) is the static stiffness matrix.
Identifying (26) as a Ricatti equation, it can be solved for \([K]^{-1}\) with the introduction of the Hamiltonian matrix \([Z]\) defined as

\[
Z = \begin{bmatrix}
(E^\top)^{-1} (E^\top)^{-1} & - (E^\top)^{-1} \\
-E^2 + E^{1} (E^\top)^{-1} (E^\top)^{-1} & - E^{1} (E^\top)^{-1}
\end{bmatrix}^{-1}
\]  

(27)

Applying a real orthogonal transformation \(V\) to \(Z\)

\[
ZV = VS = V \begin{bmatrix}
S_{11} & S_{12} \\
S_{21} & S_{22}
\end{bmatrix}
\]  

(28)

and rearranging \(S\) in a way that the real parts of the eigenvalues of \(S_{11}\) are negative and that of \(S_{22}\) are positive and partitioning conformably \(V\) as

\[
V = \begin{bmatrix}
V_{11} & V_{12} \\
V_{21} & V_{22}
\end{bmatrix}
\]  

(29)

the matrix solution of eq. (25) will be

\[
K = V_{21} (V_{11})^{-1}
\]  

(30)

To determine the mass matrix \(M\), it is convenient to express the dynamic stiffness matrix as

\[
S(\omega) = K - \omega^2 M
\]  

(31)

Substituting (31) in (25), an approximation of \(M\) for low frequencies can be obtained from:

\[
\left(-K + E^{1}\right) \left(E^{1}\right)^{-1} - I \right) \left(M + M \left(E^{1}\right)^{-1} \left(-K + (E^{1})^\top\right) - I\right) + M^\top = 0
\]  

(32)

Equation (32) can be solved as a Lyapunov equation. Alternatively, the mass matrix can be determined through the previous results used to obtain \(K\). Post multiplying (28) by \(V_{11}\) leads to

\[
(E^\top)^{-1} \left(-K + (E^{1})^\top\right) - I = V_{11} S_{11} (V_{11})^{-1}
\]  

(33)

Substituting (33) in (32) and pre-multiplying by \((V_{11})^\top\) and post-multiplying by \(V_{11}\) results in

\[
(I + (S_{11})^\top) m + m (I - S_{11}) = (V_{11})^\top M^\top V_{11}
\]  

(34)

where

\[
m = (V_{11})^\top M V_{11}
\]  

(35)

Using back substitution (34) can be solved for \(m\) and the mass matrix is obtained from (35) as

\[
M = \left(V_{11}\right)^{-1} \left(V_{11}\right)^{-1} m
\]  

(36)

6. MODAL PARAMETERS ESTIMATION

Modal parameters can be determined by solving the dynamic eigenproblem equation

\[
(K - \omega^2 M) \Phi = 0
\]  

(37)

which produces the eigenvalues (squared natural frequencies), \(\omega^2\), and eigenvectors (mode shapes), \(\Phi\).

As will be shown in the numerical example, the use of equation (37) with the mass matrix from equation (36) is valid only for very low frequency ranges. As the frequency range increase, the accuracy of the last natural frequencies calculated decrease severely. This precision loss becomes this solution inadequate to structural analysis applications.

In order to overcome this problem the authors suggest another approach to obtain more accurate natural frequencies in a larger frequency range. The method is based on the system FRFs obtained directly from the SBFEM dynamic stiffness matrix \(S(\omega)\) (Eq. 25), instead of \(M\) and \(K\) matrices. By re-arranging equation (25) we obtain

\[
\tilde{S}(\omega) = \frac{1}{\omega} \left(\omega^2 - (S(\omega) - E^{1}) (E^{1})^{-1} \left(S(\omega) - (E^{1})^\top\right)- \omega^2 M^\top \right)
\]  

(38)

Equation (38) is a Ricatti first order differential equation. Then, to obtain \(S(\omega)\) a numerical solution based on a fourth order Hunge-Kuta integration method was used.

The frequency response functions matrix between the displacement response points and the force excitation points can be obtained from

\[
H(\omega) = \frac{1}{K - \omega^2 M}
\]  

(39)

By substituting equation (31) in equation (39), the FRF matrix will be

\[
H(\omega) = (S(\omega))^{-1}
\]  

(40)

From this numerical FRFs using SBFEM, it is possible to extract the modal parameters by using well-known modal parameter identification methods (Mesquita Neto et al., [4]).

The features and application results of both discussed approaches are presented in the next section.

7. NUMERICAL EXAMPLE

In this section the vibration of a clamped-free plate in plane stress (fig. 2), as presented by Nardini & Brebbia [5], is examined. Adopting the dimension rate \(L/d = 4\), \(E/\rho = 10^4\) (Young’s modulus \(E\), and mass density \(\rho\) and Poisson’s ratio \(\nu = 0.2\), the natural frequencies are determined by SBFEM using the eigenproblem solution approach, Eq. (30), (35), and (37). For several boundary discretization, the natural frequencies are obtained and compared with that from FEM (Table 1). It is clear that a good agreement is obtained only for the first frequencies.
Better results are obtained by using the FRFs through the dynamic stiffness matrix, Eq. (38). The natural frequencies are extracted by performing a theoretical modal analysis scheme using four FRFs. In fig. 3 it is presented FRFs from SBFEM and FEM with excitation at point P direction-\( y \), and displacement response at the same point and direction, and in
In table II it is compared the extracted frequencies from SBFEM and the calculated values from FEM. It can be observed a good agreement between them.

<table>
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<th>SBFEM</th>
<th>FEM</th>
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<tbody>
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<tr>
<td>5.1741</td>
<td>5.1914</td>
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</table>

8. FINAL REMARKS

A review of plane stress elastodynamics scaled boundary finite element method formulation was presented, and applied to obtain the FRFs and the natural frequencies for in-plane motion of a plate. The results shown that the solution using static stiffness and mass matrices do not produces accurate natural frequency results for the whole frequency band, however the use of the dynamic stiffness matrix was proven capable to determine the FRFs from which the natural frequencies could be extracted more precisely. Due to the singularities of the dynamic equation derivatives, the numerical determination of the FRFs was very time consuming. It was shown that the SBFEM is suitable to model the dynamic behavior of plates and to obtain their FRFs, been an alternative to the traditional numerical methods, with the advantages that are usual for boundary methods.

REFERENCES