Norm–Based Approximation in Invex Multi-Objective Programming Problems

Tarek Emam
Dept. of Mathematics, Faculty of Science, Suez University, Egypt.
Dept. of Mathematics, Faculty of Science, University of Hail, KSA.

Abstract— This paper addresses the problem of capturing nondominated points on convex Pareto frontiers, which are encountered in invex multi-objective programming problems. An algorithm to find a piecewise linear approximation of the nondominated set of convex Pareto frontier are applied.

Index Term— Approximation, Nondominated points, Invex multi-objective problems, Block norms.

1 INTRODUCTION
In multi-objective programming, several conflicting and non-commensurate objective functions have to be optimized over a feasible set determined by constraint functions. Due to the conflicting nature of the objectives, a unique feasible solution optimizing all the objectives does not exist. Based on the commonly used Pareto concept of optimality, one has to deal with a rather large or infinite number of efficient solutions. In most works, an assumption of convexity was made for the objective functions. Very recently, some generalization convexity has received more attention (see, for example, [1, 3, 6, 7]). A significant generalization of convex functions is invex function introduced first by Hanson [4], which has greatly been applied in nonlinear programming and other branches of pure and applied sciences.

Since there are infinitely many efficient solutions, an approximated description of the solution set becomes an appealing alternative. The approximation algorithm proposed in this paper follows earlier researches initiated by Schandl [9] and continued by Schandl et al. [10]. The approximation comes in the form of a polyhedral distance measure that is being constructed successively during the execution of the algorithm. The measure is being utilized both to evaluate the quality of the approximation and to generate additionally nondominated solutions.

Inspired and motivated by above works, the purpose of this paper is to generate nondominated solutions of invex multi-objective programming problems. The outline of the paper is as follows: In the next section, we mention some mathematical preliminaries that have an important role not only in traditional programming but also in multi-objective programming. In section 3, we state the invex multi-objective programming problem and extend invex functions to the so-called cone invex functions and derive some results about it. The oblique norms are defined and a theoretical basis for the approximation algorithm is discussed in section 4. Finally, an approximation approach developed in [9] is applied to an invex multi-objective programming problem in section 5.

2 MATHEMATICAL PRELIMINARIES
To facilitate further discussions, the following notation is used throughout thesis. Let $u, v \in \mathbb{R}^n$ be two vectors.
1. We denote components of vectors by subscripts and enumerate vectors by superscripts.
2. $u < v$ denotes $u_i < v_i$ for all $i = 1, 2, ..., n$. $u \leq v$ denotes $u_i \leq v_i$ for all $i = 1, 2, ..., n$, but $u \neq v$. $u \leq v$ allows equality. The symbols $<, \leq, >$ are used accordingly.
3. Let $R^n_\geq = \{x \in \mathbb{R}^n : x \geq 0\}$. If $S \subseteq \mathbb{R}^n$, then $S_\geq = S \cap R^n_\geq$. The sets $R^n_\leq, R^n_\leftrightarrow, S_\geq$ and $S_\leq$ are defined accordingly. In the following, we recall some general definitions and notations.

Definition 1 (Cone and convex cone) [2] A subset $M$ of $\mathbb{R}^n$ is called a cone if $\lambda x \in M$ whenever $x \in M$ and $\lambda > 0$. Moreover, a cone $M$ is said to be a convex cone when it is also convex.

Definition 2 (Cone Convexity) [2]
Given a set $M$ and a convex cone $D$ in $\mathbb{R}^n$, $M$ is said to be $D$-convex if $M + D$ is a convex set.

Definition 3 (Convex cone function) [2] Let $M$ be a convex set in $\mathbb{R}^n$, $f$ be a function from $M$ into $\mathbb{R}^k$, and $D$ be a convex cone in $\mathbb{R}^k$. Then $f$ is said to be $D$-convex if for any $x^1, x^2 \in M$ and for any $\lambda \in [0, 1], 
\lambda f(x^1) + (1 - \lambda)f(x^2) - f(\lambda x^1 + (1 - \lambda)x^2) \in D$.

Definition 4 (Invex Set) [4]
The set $M \subseteq \mathbb{R}^n$ is said to be invex at $y \in M$ with respect to a map $\eta : M \times M \to \mathbb{R}^n$, if $y + \lambda \eta(x, y) \in M$, for all $x \in M$ and $\lambda \in [0, 1]$. $M$ is said to be invex set with respect to $\eta$, if it is invex at each $y \in M$ with respect to the same $\eta$. 

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Definition 5 (Invex Function) [4]
Let $M \subseteq \mathbb{R}^n$ be an invex set with respect to a map $\eta : M \times M \to \mathbb{R}^n$. The function $f : M \to \mathbb{R}^n$ is said to be invex at $y \in M$ with respect to $\eta$, if
\[ f(y + \lambda \eta(x,y)) \leq \lambda f(x) + (1-\lambda) f(y), \]
for all $x \in M$ and $\lambda \in [0,1]$.

Definition 6 (Domination structure) [2]
For each $z \in Z \subseteq \mathbb{R}^k$, we define the set of domination factors
\[ D(z) = \{ d \in \mathbb{R}^k : z > z + d \} \cup \{0\}. \]
This means that deviation $d \in D(z)$ from $z$ is less preferred to the original $z$. Then the point to set map $D$ from $Z$ to $\mathbb{R}^k$ clearly represent the given preference order. We call $D$ the domination structure.

Definition 7 (Nondominated set) [2]
Given a set $Z$ in $\mathbb{R}^k$ and a domination structure $D(\cdot)$, the set of nondominated elements is defined
\[ N(Z,D) = \{ \tilde{z} \in Z : \text{there is no } z \neq \tilde{z} \in Z \text{ s.t. } \tilde{z} \in z + D(z) \}, \]
and is called the nondominated set.

3 Problem Formulation
Let $f : \mathbb{R}^n \to \mathbb{R}^k$ and $g : \mathbb{R}^n \to \mathbb{R}^n$ be invex functions with respect to a map $\eta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$. An invex multi-objective programming problem is formulated as follows:
\[
\begin{align*}
\text{Min} & \quad f(x), \\
\text{subject to} & \quad x \in M = \{ x \in \mathbb{R}^n : g(x) \leq 0 \}.
\end{align*}
\]
In the objective space $\mathbb{R}^k$, for problem (P), the set of all feasible criterion vectors is as follows:
\[ Z = f(M) = \{ z : z = f(x), x \in M \}. \]
The most fundamental kind of efficient solution is obtained when $D = \mathbb{R}^k_+$ and is usually called a Pareto solution or Noninferior solution.

Definition 8 (efficient solution) [2]
A point $x^* \in M$ for (P) is be an efficient solution to the problem (P) if there is no $x \in M$ such that $f(x) \leq f(x^*)$. The set of all nondominated points $N$ and the set of all efficient points $F$ of problem (P) are defined as follows:
\[ N = \{ z \in Z : \exists \tilde{z} \text{ s.t. } \tilde{z} \leq z \}, \]
and
\[ F = \{ x \in M : f(x) \in N \}. \]

Definition 9 (Geoffrion’s proper efficiency) [2]
A point $z \in N$ is called properly nondominated, if there exists a scalar $q > 0$ such that for each $i, i = 1, 2, \ldots, k$, and each $z \in Z$ satisfying $z^i > z^i$, there exists at least one $j \neq i$ with $z^j > z^j$ and $\frac{z^j - z^j}{z^j - z^j} \leq q$. Otherwise $z \in N$ is called improperly nondominated. The set of all properly points is denoted by $N_p$.

Now, we extend the concept of invexity to the cone invexity which enables us to deal with invex multi-objective programming problems.

Definition 10 (Cone Invex Function)
Let $M \subseteq \mathbb{R}^n$ be an invex set with respect to a map $\eta : M \times M \to \mathbb{R}^n$, $f$ be a function from M into $\mathbb{R}^k$, and $D$ be a convex cone in $\mathbb{R}^k$. Then, $f$ is said to be $D$-invex with respect to $\eta$, if
\[ \lambda f(x) + (1-\lambda) f(y) - f(y + \lambda \eta(x,y)) \in D \text{ for all } x, y \in M \text{ and } \lambda \in [0,1]. \]

Remark 1
Let $M \subseteq \mathbb{R}^n$ be an invex set with respect to a map $\eta : M \times M \to \mathbb{R}^n$. A function $f : M \to \mathbb{R}^k$ is $D$-invex with respect to $\eta$ if and only if $f$ is $R^k_+$-invex with respect to $\eta$.

Proposition 1
Let $M \subseteq \mathbb{R}^n$ be an invex set with respect to a map $\eta : M \times M \to \mathbb{R}^n$ and $D$ be a convex cone in $\mathbb{R}^k$. If $f : M \to \mathbb{R}^k$ is $D$-invex with respect to $\eta$, then the set $f(M)$ is $D$-convex set.

Proof. Let $x, y$ be any two points in the set $f(M) + D$, then there exist $z^1, z^2 \in M$ and $d^1, d^2 \in D$ such that
\[ x = f(z^1) + d^1, y = f(z^2) + d^2. \]
For $\lambda \in [0,1]$, we have
\[ \lambda x + (1-\lambda) y = \lambda f(z^1) + d^1 + (1-\lambda) [f(z^2) + d^2] = \lambda d^1 + (1-\lambda) d^2 + \lambda f(z^1) + (1-\lambda) f(z^2) + \lambda \eta(z^1, z^2) - f(z^1 + \lambda \eta(z^1, z^2)). \]

invexity of $M$ and $D$-invexity of $f$, we get
\lambda x + (1 - \lambda) y = f(z^2 + \lambda \eta(z^2)) + \{\lambda d^1 + (1 - \lambda) d^2 + \lambda \eta(z^2) - f(z^2) + \lambda \eta(z^2)} \in f(M) + D.

which mean that \( f(M) + D \) is convex set and hence \( f(M) \) is D-convex set.

We assume that the set \( Z \) is \( R^k \)-closed and that we can find \( u \in R^k \) so that \( u + Z \subseteq R^k \). The point \( \hat{z} \in R^k \) with \( \hat{z}_i = \min \{ f(x) : x \in M \} - \epsilon_i \), \( i = 1, 2, \ldots, k \) is called the ideal (utopia) criterion vector, where the components of \( \epsilon = (\epsilon_1, \epsilon_2, \ldots, \epsilon_k) \subseteq R^k \) are small positive numbers.

The point \( z^* \subseteq R^2 \) with \( z_i^* = \min \{ z_j : z_i = \min z \} \), \( i = 1, 2 \) is called the nadir point. This definition cannot be easily generalized to more than two dimensions. Define the set \( Z^* = \{ z^*_i \} \). Then the \( i^{th} \) component, \( i = 1, 2, \ldots, k \) of the default reference point is defined as \( z_i^* = \max \{ z_i : z_i \in Z^* \} \).

This is a possible generalization of the nadir point concept, can be calculated relatively easily and also provides the utopia point

4 Oblique Norms in Multi-objective Programming

The concept of oblique norms was introduced in Schandl et al. [11] and Schandl [9]. Since oblique norms can be viewed as a special class of block norms, we first review some basic definitions about block norms and more general, polyhedral gauges. Then oblique norms are discussed in the context of multi-objective programming.

Definition 11 [10]

Let \( B \subseteq R^k \) be a convex and compact set in \( R^k \) containing the origin in its interior and let \( x \in R^k \). The gauge \( \gamma \) of \( x \) with respect to \( B \) is then defined as

\( \gamma(x) = \min \{ \lambda \geq 0 : x \in \lambda B \} \).

Given a gauge \( \gamma \), the set

\( B = \{ x \in R^k : \gamma(x) \leq 1 \} \)

is called unit ball or sublevel set of level 1.

Definition 12 [10]

If the set \( B \) is a convex polytope, then \( \gamma \) is called a polyhedral gauge and is sometimes referred to as \( \gamma_k \). Let

\( \text{ext}(B) = \{ v^1, v^2, \ldots, v^k \} \)

be the set of extreme points of \( B \); \( v^1, v^2, \ldots, v^k \) are called fundamental vectors. The half-lines \( d^1, d^2, \ldots, d^k \) starting at the origin and passing through the extreme points \( v^1, v^2, \ldots, v^k \) are called fundamental directions.

If \( B \) is symmetric with respect to the origin, then it is called a block norm.

Definition 13

The fundamental vectors defined by the extreme points of a facet of \( B \) span a fundamental cone. The cone spanned by the fundamental vectors \( v^i \) and \( v^j \) are referred to as \( C(v^i, v^j) \). If \( x \) is in a fundamental cone \( C \) of polyhedral gauge \( \gamma \) then one needs to consider only the fundamental vectors generating this cone to calculate the gauge of \( x \). This result was proven in Schandl [10] for the multi-objective case.

Theorem 1 [10]

Let \( \gamma \) be a polyhedral gauge with unit ball \( B \subseteq R^k \). Let \( \bar{z} \subseteq C \) where \( C \) is the fundamental cone generated by \( v^1, v^2, \ldots, v^l, l \geq k \). Let \( \bar{z} = \sum_{i=1}^l \lambda_i v^i \) be a representation of \( \bar{z} \) in terms of \( v^1, v^2, \ldots, v^l \). Then \( \gamma(\bar{z}) = \sum_{i=1}^l \lambda_i \).

Not that all representations \( \bar{z} = \sum_{i=1}^l \lambda_i v^i \) can be used to calculate \( \gamma(\bar{z}) \), even combinations where one or more \( \lambda_i \) are negative which is only possible if \( l > k \). If \( C \) is generated by \( k \) fundamental vectors though, the representation of \( \bar{z} \) in term \( v^1, v^2, \ldots, v^k \) is unique and all corresponding \( \lambda_i \) all corresponding \( \lambda_i \) are nonnegative.

For the definition of oblique norm we additionally need the concept of sets. Let \( u \in R^k \). The reflection set of \( u \) is defined as \( R(u) = \{ w \in R^k : |w_i| = |u_i| \forall i = 1, 2, \ldots, k \} \).

Definition 14

A block norm \( \gamma \) with a unit ball \( B \) is called oblique if

(i) \( \gamma(w) = \gamma(u) \forall w \in R(u), u \in R^k \), and

(ii) \( (z - R^k) \cap R^k \cap \partial B = \{ z \} \forall z \in (\partial B) \).

Observe that an oblique norm is a block norm where no facet of the unit ball is parallel to any coordinate axis. Moreover, the structure of the norm’s unit ball is the same in each orthon of the coordinate system. This property is convenient for the generation of nondominated solutions of (P) since they may only occur in \( \bar{z} + R^k \). An example of an oblique norm in \( \bar{R}^k \) is given in Figure 1.
Schandl et al. [10] show that oblique norms are effective tools to generate nondominated solutions of general multi-objective programs. In particular, we examine the relationship between properly nondominated solutions of invex multi-objective programming problems, and optimal solutions of their scalarization by means of an oblique norm. The following two theorems justify the application of oblique norm for the generation of nondominated solutions.

**Theorem 2** [10]
Assume without loss of generality that $0 \in Z + R^k$. Let $\gamma$ be an oblique norm with the unit ball $B$. If $\bar{z} \in R^k$ is a solution of
$$\max \gamma(z) \quad \text{s.t.} \quad z \in -R^k \cap Z. \quad (P_\gamma)$$
Then $\bar{z}$ is nondominated.

Unfortunately, we cannot guarantee to find all nondominated points using an oblique norm with its unit ball’s center in $Z + R^k$ in the general setting of Theorem 2. Therefore, the next Theorem is proved for the invex multi-objective programming problems.

**Theorem 3**
Let $M \subseteq R^n$ be an invex set with respect to a map $\eta: M \times M \to R^n$ and $f: M \to R^k$ be an invex function with respect to some $\eta$. If $\bar{z}$ is properly nondominated with $\bar{z} \in -R^k \cap N_\rho$, then there exists an oblique norm $\gamma$ so that $\bar{z}$ solves the problem $\max \gamma(z) \quad \text{s.t.} \quad z \in -R^k \cap Z \quad (P_\gamma)$

**Proof.** Since $f: M \to R^k$ is an invex function with respect to $\eta: M \times M \to R^n$, then $f$ is $R^k$-invex with respect to $\eta$ (see Remark 1). By using Proposition 1, we conclude that $Z$ is $R^k$-convex set. $R^k$-convexity of $Z$ implies that there exists an oblique norm $\gamma$ such that $\bar{z}$ solves the problem $P_\gamma$ (see Theorem 3.6 of [10]).

5 DESCRIPTION OF APPROXIMATION APPROACH
For invex multi-objective programs, an approximation algorithm based on Theorem 2 and 3 can be designed that utilizes oblique norms for the generation of nondominated solutions. To keep explanations straight-forward, the general idea of this approach will be outlined using a bicriteria example problem. The approximation process is started by choosing a reference point $z^0 \in Z + R^k$ and defining $z^0 - R^k$ as the region in which the nondominated set $N$ is approximated. This might be a currently implemented (not nondominated) solution or just a (not necessarily feasible) guess. A first approximation is obtained by exploring the feasible set along $l \geq k$ search directions $d_1, d_2, ..., d_l \in -R^k$, specified by the decision maker. To obtain nondominated points along these search directions, an adaptation of the direction method introduced in Pascoletti and Serafini [8] is modified in Schandl [9]. In the example given in Figure (2-a), the search directions are chosen as the negative unit vectors in $R^2$, $d_1 = (-1, 0)$ and $d_2 = (0, -1)$ yielding the points $z^1$ and $z^2$. These two points together with the reference point $z^0$ are used to define a cone and a first approximation, see Figure (2-b). Interpreting this approximation as the lower left part of the unit ball of an oblique norm $\gamma$ (or, more general, of a polyhedral gauge) with $z^0$ as its center, this norm is then maximized in $Z \cap (z^0 + R^2)$. Consequently the next point $z^3$ in the problem) is found as a solution of problem $(P_\gamma)$, where $\gamma$ is an oblique norm (gauge), see Figure (2-c).

The point $z^3$ is added to the approximation by building the convex hull of the points generated so far and thus updating the approximation and the underlying norm (gauge) simultaneously as shown in Figure (2-d). Continuing this process, we get a finer approximation of nondominated set while generating of nondominated points and updating the unit ball of the oblique norm (gauge), see Figures (2-e) and (2-f). In each iteration, the point of maximal norm (gauge) is added. Since this point is "farthest away" from the approximation with respect to the current oblique norm (gauge), we always add the point of worst approximation with respect to this norm (gauge).

There are two possible stopping criteria; usually, at least one of them must given. The first one is an upper bound $\varepsilon > 0$ on the maximal deviation such that $\text{dev}(z) = |\|z\| - 1|$. As soon as we get $\text{dev}(\bar{z}) < \varepsilon$ for a point that should be added next, the algorithm stops. The other possibility is to give an integer $\maxConeNo \geq 1$, which specified the maximum number of cones to be generated. The main loop of the algorithm continues until one of possible stopping criteria is satisfied. At the end of the loop the sorted list of $r$ nondominated points is printed and can be used to visualize the approximated set.

Observe that in each iteration the maximization problem $(P_\gamma)$ has to be has to be solved only in those cones whose facets were newly generated due to the addition of the last point. This includes new and modified cones. By updating the convex hull, the resulting approximation is always $R^k$-convex. Schandl et al. [10] present the following Theorem which shows that the quality of the approximation improves...
with each new point if we assume that $Z$ is $R^k_+$-convex.

**Theorem 4** [10] Let $Z \subseteq R^k$ be $R^k_+$-convex and $\gamma^q$ be an approximating oblique norm (oblique gauge) constructed from $q$ nondominated points, or points on the boundary of $Z$. Let $z_0$ be the solution of

$$\max \gamma^q(z) \quad \text{s.t.} \quad z \in Z \cap (z^0 - R^k_+)$$

Let $\gamma^{q+1}$ be the updated norm (gauge) including the new point $z^k$. Then $\gamma^{q+1}(z) \leq \gamma^q(z)$ $\forall z \in Z \cap (z^0 - R^k_+)$.

**Example 1**

This example consists of two objective functions, which are minimized subject to two inequality constraints. This problem was presented by Jia-Wei, C.; et al. [5]. Let $\eta : R^2 \times R^2 \rightarrow R^2$ be defined as

$$\eta(x, y) = (2^{x_2/y_2} (x_1 - y_1), x_2 - \pi^{x_2/y_2}).$$

Consider the invex bicriteria programming problem

$$\min f_1(x_1, x_2) = 2x_1, \quad (x_1, x_2) \in M = \{(x_1, x_2) \in R^2 : \min f_2(x_1, x_2) = -3x_2^2, \quad \text{s.t.} \quad -5x_1 \leq x_2, x_1 \leq 0\},$$

where $M$, $f_1$, and $f_2$ are invex with respect to $\eta$. It is clear that, $Z$ is $R^2_+$-convex set. By using the gauge method on $Z$, we get the approximated nondominated set $N = \{(z_1, z_2) \in Z : z_1 = -10, \text{or} \ z_2 = 0\}$, and hence the efficient set is

$$F = \{(x_1, x_2) \in M : x_1 = -5, \text{or} \ x_2 = 0\}.$$

**REFERENCES**