Knot Insertion and Reparametrization of Interval B-spline Curves

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Abstract—Knot insertion is the operation of obtaining a new representation of a B-spline curve by introducing additional knot values to the defining knot vector. The new curve has control points consisting of the original control points and additional new control points corresponding to the number of new knot values. So knot insertions give additional control points which provide extra shape control without necessarily subdividing the curve. However, if following a knot insertion operation a knot has multiplicity equal to the degree, then the B-spline is split into two B-splines at that knot value. In this paper the concept of knot insertion for analyzing interval B-spline curve has been introduced. The four fixed Kharitonov’s polynomials (four fixed B-spline curves) associated with the original interval B-spline curve are obtained. The four fixed Kharitonov’s polynomials (four fixed B-spline curves) are subdivided by inserting additional knot values while maintaining an open uniform knot vector. Finally, the required interval control points are obtained from the fixed control points of the four fixed subdivided Kharitonov’s polynomials. The problem of parametric interval B-spline curve reparametrization is also discussed. The shape of the curve remains unchanged during the process of reparametrization; only the way the curve is described is altered. If it is important that the degree of the given curve should be kept unchanged, we may choose a linear reparametrization function. Numerical examples are included in order to demonstrate the effectiveness of the proposed method.

Index Terms—Knot insertion, reparametrization, interval B-spline curve, image processing, CAGD.

I. INTRODUCTION

Parametric representation for curves is important in computer-aided geometric design, medical imaging, computer vision, computer graphics, shape matching, and face/object recognition. They are far better alternatives to free form representation, which are plagued with unboundedness and stability problems. Parametric representations are widely used since they allow considerable flexibility for shaping and design. A curve that actually passes through each control point is called an interpolating curve; a curve that passes near to the control points but not necessarily through them is called an approximating curve. B-spline curve is among the most commonly used method for curve and surface design, and it has been widely used in practical CAD systems.

B-splines are like Bezier curves because they both use a control polygon to define the curve, and are helpful due to their control points’ local control of the resulting shape. The B in B-spline stands for “basis”, and the basis is specified by the Cox-de Boor formula for computing the basis function. What uniquely sets them apart from Bezier curves is that a vector of scalars called a knot vector is figured in to the computation of the basis functions.

When these knots are spaced evenly, the B-spline is said to be uniform, and non-uniform otherwise. The basis function considers the knot vector in every computation.

An interval B-spline curve is a B-spline curve whose control points are rectangles (the sides of which are parallel to coordinate axis) in a plane. Such a representation of parametric curves can account for error tolerances. Based upon the interval representation of parametric curves and surfaces, robust algorithms for many geometric operations such as curve/curve intersection were proposed [1]. The series of works by the authors of [1] indicate that using interval arithmetic will substantially increase the numerical stability in geometric computations and thus enhance the robustness of current CAD/CAM systems.

An interval B-spline curve is uniquely defined by its degree, interval control points and knot values. The modification of a curve plays a central role in CAD systems, hence numerous methods are presented to control the shape of a fixed B-spline curve by modifying one of its data mentioned above.

Knot insertion is one of the most useful and powerful procedures for analyzing B-spline curves and surfaces. As a constructive means for subdividing curves and surfaces, it is an important practical technique. Subdivision is useful also in rendering and intersection algorithms. Knot insertion is a vital theoretical tool, it can be even be used as the primary approach in the development of B-spline curve theory and geometric modeling. Furthermore, knot insertion is a simple tool, both computationally inexpensive and conceptually straightforward.

Parametrization aids computation in the sense that it provides a built-in parameter space for direct evaluation of quantities like tangents, normal, surface/plane intersects and projections. Same curve can be represented by multiple parametrizations. Hence, in free form design reparametrization can be used to reconcile parametrization of different curve segments (or surface patches) that have been defined independently. Reparametrization of a curve means to change how a curve is parametrized, i.e., to change which parameter value is assigned to each point on the curve. Reparametrization can be performed by a parameter substitution.

Parametric interval B-spline curves are interactive. It is possible to control the shape of the interval B-spline curve by moving the interval control points and by smoothly connecting individual interval segments. Imagine a situation where the interval points are moved and maneuvered for a while, but the interval curve “refuses” to get the right shape. This indicates that there are not enough interval points. There are two ways to increase the number of interval points.

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One is to add an interval point to a segment while increasing its degree. This is called degree elevation [2], [3], [4], [5]. An alternative is to subdivide an interval B-spline curve segment into two interval segments such that there is no change in the shape of the curve [6].

In curve design, the problem often is how to balance the desire for constructing a particular shape for a curve and obtaining a proper parametrization. Most often, we may construct the initial curve to interpolate/approximate the given data points with an initial parametrization using one of the various known techniques. However, the curves are refined to achieve the desired shape by various modifications of weights and/or control points. But, our parametrization is lost! That is the small changes in curve shape might lead to a bad or improper parametrization, which if used to construct surfaces results in badly parametrized surfaces. Hence, it is necessary to reparametrize the curve/surface to correct such situations where the shape is right and the parametrization is wrong.

This paper is organized as follows. Section II contains knot insertion of interval B-spline curves, whereas section III provides reparametrization of interval B-spline curves, while section IV shows numerical examples, and the final section offers conclusions.

II. KNOT INSERTION OF INTERVAL B-SPLINE CURVES

There are two basic techniques for increasing the flexibility of a B-spline curve, degree elevation and knot insertion (subdivision). One of the advantages of degree elevation is that the B-spline curve remains infinitely differentiable, whereas subdivision reduces the differentiability at the inserted knots. The loss of differentiability at the new knots depends on the multiplicity of the knot values. For simple knot vectors with no multiple internal knot values, the differentiability reduces to \( C^{k-2} \) at the knot values. Early degree elevation algorithms were developed in [7] and [8]. More efficient algorithms are given in [9], [10] and [11].

The flexibility of a Bezier curve is increased by raising the degree of the polynomial curve by adding an additional vertex to the control polygon. The flexibility of a B-spline curve is also increased by raising the order of the B-spline basis and hence of the polynomial segments. [7] provide both the theory and an algorithm for degree elevation of B-spline curves, as an alternative to degree elevation, the flexibility of a B-spline curve is increased by inserting additional knot values into the knot vector. Inserting a single knot value is referred to as knot insertion. Inserting multiple knot values is called knot refinement. The effect is to locally split a piecewise polynomial segment for a given knot value interval (parametric interval) into two piecewise polynomial segments over that interval. There are two basic methods for complying knot value insertion. The first is the so-called Oslo algorithm developed in [12] and the one developed in [13], which simultaneously insert multiple knot values into the knot vector. The second method [14] and [15], sequentially inserts single knot values into the knot vector.

The basic idea behind either degree elevation or knot insertion is to increase the flexibility of the curve (or surface) basis, and hence of the curve, without changing the shape of the curve (or surface). The success of the idea depends on the fact that there are an infinite number of control polygons with more than the minimum number of vertices that represent identical B-spline curves. Subsequent manipulation of the new control polygon vertices is used to change the curve shape.

Let \( P^i(u) \) be the position vector along the interval curve as a function of the parameter \( u \), an interval B-spline curve is given by:

\[
P^i(u) = \sum_{i=1}^{n+1} [p_i^L, p_i^R] N_{i,k}(u)
\]

\[
u_{\text{min}} \leq u < u_{\text{max}} \quad \text{and} \quad 2 \leq k \leq n + 1
\]

with knot vector \( U = \{u_1, u_2, \ldots, u_{n+k+1}\} \), where the \( [p_i^L, p_i^R] \) for \( i = 1, 2, \ldots, n + 1 \) are the interval position vectors of the \( (n + 1) \) interval control polygon vertices, and the \( N_{i,k} \) are the normalized B-spline basis functions. For the \( ith \) normalized B-spline basis function of order \( k \) (degree \( k-1 \)), the basis functions \( N_{i,k}(u) \) are defined by the Cox-de Boor recursion formulas. Specifically:

\[
N_{i,1}(u) = \begin{cases} 
1, & \text{if } u_i \leq u \leq u_{i+1} \\
0, & \text{otherwise}
\end{cases}
\]

\[
N_{i,k}(u) = \frac{(u - u_i)N_{i,k-1}(u)}{u_{i+k-1} - u_i} + \frac{(u_{i+k} - u)N_{i+1,k-1}(u)}{u_{i+k} - u_{i+1}}
\]

The values of \( u_i \) are elements of a knot vector satisfying the relation \( u_i < u_{i+1} \). The parameter \( u \) varies from \( u_{\text{min}} \) to \( u_{\text{max}} \) along the interval curve \( P^i(u) \). The convention \( (0/0 = 0) \) is adopted. Formally, a B-spline curve is defined as a polynomial spline function of order \( k \) (degree \( k-1 \)), because it satisfies the following two conditions: (1) \( P^i(u) \) is an interval polynomial of degree \( (k-1) \) on each interval \( u_i \leq u < u_{i+1} \). (2) \( P^i(u) \) and its derivatives of order \( (1, 2, \ldots, k-2) \) are all continuous over the entire interval curve.

Note that \( u_i \) are called knots in equation (2), which are defined as:

\[
u_i = \begin{cases}
0, & i < k \\
i - k + 1, & k \leq i \leq n \\
n - k + 2, & i > n
\end{cases}
\]

After knot insertion, the new curve is

\[
Q^k(\bar{u})
\]

defined by:
with the new knot vector \( \bar{U} = \{ \bar{u}_1, \bar{u}_2, \ldots, \bar{u}_{m+k+1} \} \), where \((m > n)\). The objective is to determine the new interval control polygon vertices, \([q_{i}^{-}, q_{i}^{+}]\) such that \( P^j(u) = Q^j(\bar{u}) \).

The four fixed Kharitonov's polynomials (four fixed B-spline curves)\([16]\) associated with the original interval B-spline curve \( P^j(u) \) are:

\[
P^j(u) = \sum_{i=1}^{n+1} a^i_j \tilde{N}_{i,j}(u) \quad (j = 1, 2, 3, 4)
\]

with knot vector \( U = \{ u_1, u_2, \ldots, u_{n+k+1} \} \)

The four fixed Kharitonov's polynomials (four fixed B-spline curves) associated with the original interval B-spline curve \( P^j(u) \) can be written as follows:

\[
Q^j(\bar{u}) = \sum_{j=1}^{m+1} [q_{j}^{-}, q_{j}^{+}] M_{j,k}(\bar{u})
\]

(4)

with the new knot vector \( \bar{U} = \{ \bar{u}_1, \bar{u}_2, \ldots, \bar{u}_{m+k+1} \} \), where \((m > n)\).

Now, the problem can be converted into: determine the new control polygon vertices of the four fixed Kharitonov's polynomials (four fixed B-spline curves) \( Q^j_m(\bar{u}) \) that correspond the four fixed Kharitonov's polynomials (four fixed B-spline curves) \( P^j_n(u) \) for \((j = 1, 2, 3, 4)\), respectively, such that:

\[
P^j_n(u) = Q^j_m(\bar{u}), \quad (j = 1, 2, 3, 4)
\]

(9)

By the Oslo algorithm \([13]\), the new \( \tilde{a}^j_{i,m} \) s for \((j = 1, 2, 3, 4)\) and \(1 \leq l \leq m\) are:

\[
\tilde{a}^j_{i,m} = \sum_{k=1}^{n+1} \beta^r_{k,l} a^j_{k,n} \quad 1 \leq k \leq n \quad and \quad 1 \leq l \leq m
\]

(10)

where the \( \beta^r_{k,l} \) s are given by the recursion relation:

\[
\beta^r_{k,l} = \begin{cases} 1, & \text{if } u_k \leq \bar{u}_l \leq u_{k+1} \\ 0, & \text{otherwise} \end{cases}
\]

\[
\beta^r_{k,l} = (\bar{u}_{l+r-1} - u_k) \frac{\beta^r_{k,l-1}}{u_{k+r-1} - u_k} + (u_{k+r} - \bar{u}_{l+r-1}) \frac{\beta^r_{k+1,l}}{u_{k+r} - u_{k+1}}
\]

(11)

Note that:

\[
\sum_{k=1}^{n+1} \beta^r_{k,l} = 1
\]

(12)

At first glance it appears that after insertion of a knot value, if the original knot vector was uniform, either periodic or open, then the final knot vector is nonuniform. However, a uniform knot vector is maintained by inserting multiple knot values midway in each existing nonzero interval.

Finally, the new interval control polygon vertices, \([q_{i}^{-}, q_{i}^{+}]\) can be obtained as follows:

\[
[q_{i}^{-}, q_{i}^{+}] = \left[ \min(\tilde{a}^j_{i,(m+1)}), \max(\tilde{a}^j_{i,(m+1)}) \right] \quad (i = 1, 2, \ldots, m + 1) \quad and \quad (j = 1, 2, 3, 4)
\]

(13)

III. REPARAMETERIZATION OF INTERVAL B-SPLINE CURVES

Although B-spline curves and surfaces are typically mathematically continuous when represented in a
computer, they are discretized. For example, to display a B-spline curve (or surface) on a monitor, several discrete points on the curve (or surface) are calculated and then connected by short straight line segments. A similar technique is used to convert curves (or surfaces) to numerical control codes to drive a machine tool. The results can be strange, wondrous, amusing, befuddling, or in some cases disastrous, depending on your viewpoint.

In general, if a $k$th order interval B-spline curve in equation (1), is defined on a knot vector:

$$ U = \{u_1, \ldots, u_k, \ldots, u_{n+2}, \ldots, u_{n+1+k}\} $$

and the four fixed $k$th order Kharitonov’s polynomials (four fixed B-spline curves) associated with the original interval B-spline curve $P_l^I(u)$ are obtained as given in equation (5). The reparameterized four fixed $k$th order Kharitonov’s polynomials (four fixed B-spline curves) associated with the original interval B-spline curve $P_l^I(u)$ are:

$$ p_j^n(\hat{u}) = \sum_{i=1}^{n+1} \hat{a}_{i,j} \hat{N}_{i,k}(\hat{u}) $$

(15)

is defined on a knot vector:

$$ \hat{U} = \{\hat{u}_1, \ldots, \hat{u}_k, \ldots, \hat{u}_{n+2}, \ldots, \hat{u}_{n+1+k}\} $$

(16)

where $\hat{u} = u(x)$, with $u(x)$ a $q$th degree polynomial, then the degree of the resulting four fixed B-spline curves is $p_q$, where $p$ is the degree of the original four fixed B-spline curves. Except for $q = 1$, reparameterization is another technique for raising the degree of the four fixed B-spline curves. Here, we concentrate on the linear case, i.e., $q = 1$. When $q = 1$, the relationship between the knot values in $U$ and $\hat{U}$ has the form:

$$ \hat{u} = au + b $$

(17)

where $a$ and $b$ are constants. In this special case, it can be shown that both the control polygon and the derivatives at the ends of the four fixed B-spline curves remain unchanged; i.e., $\hat{a}_{i,(n+1)} = a_i^{(n+1)}$, for $i = 1$ to $n + 1$, and $(p_j^n)'(u) = (p_j^n)(\hat{u})$ at $u = \hat{u} = 0$ and $u = \hat{u} = n + k + 1$.

IV. NUMERICAL EXAMPLES

Example 1: Consider an open third order $(k = 3)$ interval B-spline curve, initially defined by four $(n + 1 = 4)$ interval polygon vertices:

$$ [p_1^1, p_2^1] = [(0.00, 0.50) \times (0.00, 0.75)] $$

$$ [p_2^2, p_2^2] = [(1.00, 1.25) \times (1.50, 1.75)] $$

$$ [p_3^3, p_3^3] = [(2.00, 2.75) \times (1.25, 1.80)] $$

$$ [p_4^4, p_4^4] = [(3.00, 4.00) \times (0.00, 0.25)] $$

Initially, the open uniform knot vector is defined by:

$$ \hat{U} = [0, 0, 0, 1, 2, 2, 2] $$

The problem is to subdivide the curve by inserting additional knot values while maintaining an open uniform knot vector.

The two nonzero knot intervals $0 \rightarrow 1$ and $1 \rightarrow 2$ yield two piecewise parabolic segments comprising the B-spline curve. For convenience, a uniform knot vector with integer intervals is required after subdivision. Thus, the curve is reparameterized (as explained in section III) by multiplying each knot value in $\hat{U}$ by 2 to obtain:

$$ U = \begin{bmatrix} 0 & 0 & 0 & 1 & 2 & 4 & 4 & 4 \end{bmatrix} $$

The resulting curve is exactly the same. While maintaining a uniform knot vector, the original curve is subdivided by inserting knot values of 1 and 3 in the intervals $0 \rightarrow 2$ and $2 \rightarrow 4$, respectively. The new knot vector is:

$$ U = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 2 & 3 & 4 & 4 & 4 \end{bmatrix} $$

Four piecewise parabolic segments now comprise the B-spline curve. As explained in section II, the four fixed Kharitonov’s polynomials (four fixed Bezier curves) are found, and the new six fixed control polygon vertices, $\hat{a}_{i,1}^1$ for $(j = 1, 2, 3, 4)$ and $(l = 1, 2, 3, 4, 5, 6)$ of the four fixed Kharitonov’s polynomials (four fixed B-spline curves) are obtained. The recursion relations are used to get $\beta_{i,j}^k$ as follows:

$$ \beta_{i,1}^1 = \beta_{i,2}^1 = \beta_{i,3}^1 = \beta_{i,4}^1 = \beta_{i,5}^1 = \beta_{i,6}^1 = 1 $$

The nonzero second order $(k = 2)$ $\beta_{i,j}^k$ and the required nonzero third order $(k = 3)$ $\beta_{i,j}^k$ are obtained as given in table below.

<table>
<thead>
<tr>
<th>$i$</th>
<th>$\beta_{i,1}^1$</th>
<th>$\beta_{i,2}^1$</th>
<th>$\beta_{i,3}^1$</th>
<th>$\beta_{i,4}^1$</th>
<th>$\beta_{i,5}^1$</th>
<th>$\beta_{i,6}^1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$\beta_{1,1}^1 = 0$</td>
<td>$\beta_{1,2}^1 = 1$</td>
<td>$\beta_{1,3}^1 = 0$</td>
<td>$\beta_{1,4}^1 = 1$</td>
<td>$\beta_{1,5}^1$</td>
<td>$\beta_{1,6}^1$</td>
</tr>
<tr>
<td>2</td>
<td>$\beta_{2,1}^1 = 1$</td>
<td>$\beta_{2,2}^1 = 0$</td>
<td>$\beta_{2,3}^1 = 1/2$</td>
<td>$\beta_{2,4}^1 = 1/2$</td>
<td>$\beta_{2,5}^1 = 1/4$</td>
<td>$\beta_{2,6}^1 = 1/4$</td>
</tr>
<tr>
<td>3</td>
<td>$\beta_{3,1}^1 = 1$</td>
<td>$\beta_{3,2}^1 = 0$</td>
<td>$\beta_{3,3}^1 = 1/2$</td>
<td>$\beta_{3,4}^1 = 1/2$</td>
<td>$\beta_{3,5}^1 = 0$</td>
<td>$\beta_{3,6}^1 = 0$</td>
</tr>
<tr>
<td>4</td>
<td>$\beta_{4,1}^1 = 1$</td>
<td>$\beta_{4,2}^1 = 0$</td>
<td>$\beta_{4,3}^1 = 1/2$</td>
<td>$\beta_{4,4}^1 = 1/2$</td>
<td>$\beta_{4,5}^1 = 0$</td>
<td>$\beta_{4,6}^1 = 0$</td>
</tr>
</tbody>
</table>

The new six fixed control polygon vertices, $\hat{a}_{i,1}^1$ for $(j = 1, 2, 3, 4)$ and $(l = 1, 2, 3, 4, 5, 6)$ of the four fixed Kharitonov’s polynomials (four fixed B-spline curves) are obtained as shown in table:

<table>
<thead>
<tr>
<th>$i$</th>
<th>$a_{i,1}^1$</th>
<th>$a_{i,2}^1$</th>
<th>$a_{i,3}^1$</th>
<th>$a_{i,4}^1$</th>
<th>$a_{i,5}^1$</th>
<th>$a_{i,6}^1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(0.000,0.000)</td>
<td>(0.000,0.000)</td>
<td>(0.500,0.750)</td>
<td>(0.500,0.750)</td>
<td>(0.500,0.750)</td>
<td>(0.500,0.750)</td>
</tr>
<tr>
<td>2</td>
<td>(0.500,0.750)</td>
<td>(0.500,0.750)</td>
<td>(0.500,0.750)</td>
<td>(0.500,0.750)</td>
<td>(0.500,0.750)</td>
<td>(0.500,0.750)</td>
</tr>
<tr>
<td>3</td>
<td>(1.4375,1.5750)</td>
<td>(1.6250,1.7625)</td>
<td>(1.4375,1.6250)</td>
<td>(1.4375,1.6250)</td>
<td>(1.4375,1.6250)</td>
<td>(1.4375,1.6250)</td>
</tr>
<tr>
<td>4</td>
<td>(2.3125,1.7250)</td>
<td>(2.3750,1.7875)</td>
<td>(1.8125,1.3756)</td>
<td>(1.7500,1.3125)</td>
<td>(1.7500,1.3125)</td>
<td>(1.7500,1.3125)</td>
</tr>
</tbody>
</table>
Finally, the new six interval control polygon vertices, \( \{[q_j, q'_j]\}_{j=1}^{6} \) are calculated from the six fixed control polygon vertices, \( \tilde{a}_{i,3} \) for \( (j = 1,2,3) \) and \((l = 1,2,3,4,5,6)\) of the four fixed Kharitonov's polynomials (four fixed B-spline curves) as follows:

\[
\begin{align*}
[q_1, q'_1] & = ([0.0000,0.5000] \times [0.0000,0.7500]) \\
[q_2, q'_2] & = ([0.5000,0.8750] \times [0.7500,1.2500]) \\
[q_3, q'_3] & = ([1.2500,1.6250] \times [1.4375,2.2125]) \\
[q_4, q'_4] & = ([1.7500,2.3750] \times [1.3125,1.7875]) \\
[q_5, q'_5] & = ([2.5000,3.3750] \times [0.6250,1.0250]) \\
[q_6, q'_6] & = ([3.0000,4.0000] \times [0.0000,0.2500])
\end{align*}
\]

Simulation results in Figure (1), shows the envelopes of \( Q^i(\tilde{u}) \) and \( P^i(u) \), respectively.

![Fig. 1: Original and knot inserted curve envelopes.](image)

**Example 2:** Reparameterize the third-order interval B-spline curve defined by the interval control polygon:

\[
\begin{align*}
[p_1, p'_1] & = ([0.00,0.25] \times [0.00,0.50]) \\
[p_2, p'_2] & = ([1.00,1.25] \times [1.00,1.50]) \\
[p_3, p'_3] & = ([2.00,2.25] \times [0.00,0.50]) \\
[p_4, p'_4] & = ([3.00,3.25] \times [-1.50,-1.00]) \\
[p_5, p'_5] & = ([4.00,4.25] \times [0.00,0.50])
\end{align*}
\]

and the knot vector:

\[
\tilde{U} = [0, 0, 0, 2, 2, 5, 5, 5]
\]

to have an equal number of points on the curve in the intervals [0,2] and [2,5] (i.e., find a linear reparameterization that yields equally spaced computed points on the curve in geometric space.)

The knot vector is reparameterized to yield equal open intervals, i.e., the desired new knot vector is:

\[
U = [0, 0, 0, 5/2, 5/2, 5, 5, 5]
\]

As explained in section III, and assuming the new knot vector starts at zero, the conditions on:

\[
\hat{u} = au + b
\]

are:

\[
\hat{u} = 5/2, \quad at \quad u = 2, \quad u > 0
\]

\[
\hat{u} = 5, \quad at \quad u = 5, \quad u > 0
\]

Substituting yields:

\[
5/2 = 2a + b \\
5 = 5a + b
\]

Solving gives \( a = b = 5/6 \). Hence, the linear reparameterization is:

\[
\hat{u} = 5/6 (u + 1), \quad u > 0
\]

V. CONCLUSIONS

The meaning of knot insertion is adding a new knot into the existing knot vector without changing the shape of the curve. Knot insertion is exact, and tells us how to represent exactly a given curve or surface with additional knots, moreover knot insertion introduces more data which can be used to provide piecewise linear approximations for rendering and intersection algorithms.

The concept of knot insertion for analyzing interval B-spline curve \( P^i(u) \) for \( u_{min} \leq u < u_{max} \) and \( 2 \leq k \leq n + 1 \) is introduced in this paper. The objective is to determine the new interval B-spline curve \( Q^i(\tilde{u}) \) for \( \tilde{u}_{min} \leq \tilde{u} < \tilde{u}_{max} \) by inserting additional knot values while maintaining an open uniform knot vector such that \( P^i(u) = Q^i(\tilde{u}) \). The problem is converted into determining the new control polygon vertices of the four fixed Kharitonov's polynomials (four fixed B-spline curves) \( Q^i_m(\tilde{u}) \) that correspond the four fixed Kharitonov's polynomials (four fixed B-spline curves) \( P^i_m(u) \) for \( (j = 1,2,3,4) \), respectively, such that \( P^i_m(u) = Q^i_m(\tilde{u}) \) for \( (j = 1,2,3,4) \). Finally the new interval control polygon vertices, \( \{[q_j, q'_j]\}_{j=1}^{m+1} \) are obtained from the fixed control points of the four fixed subdivided Kharitonov's polynomials. The problem of parametric interval B-spline curve reparameterization is also discussed. This process consists of changing the current parameter of a given curve with another parameter using a reparameterization function. It should be noted that the shape of the curve remains unchanged during this process; only the way the curve is described is altered. If it is important that the degree of the curve should be kept unchanged, we may choose a linear reparameterization function.
REFERENCES


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