Fuzzy Erlangian Queuing System with State – Dependent Service Rate, Balking, Reneging and Retention of Reneged customers

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Abstract-- The aim of this paper is to derive the analytical solution of fuzzy truncated Erlangian service queue with state-dependent rate, balking, reneging and retention of reneged customers \( FM / FE_r / 1 / N (\zeta, \beta) \). We obtain \( P_{n,s} \), the probabilities that there are \( n \) units in the system and the unit in the service occupies stage \( s \) \( (s = 1, 2, \ldots, r) \). We treat this queue for general values of \( r, k \) and \( N \).

Index Term-- Fuzzy queue; Membership function; Erlangian service queue; Retention of reneged customers.

1- INTRODUCTION

This paper considers the queuing system \( FM / FE_r / 1 / N \) with state – dependent service rate, balking, reneging and retention of reneged customers concepts. The Erlang distribution, denoted by \( E_r \), is a special case of the gamma distribution, is named after A.K. Erlang who pioneered queuing system theory for its application to congestion in telephone networks. The non-truncated queue: \( M / E_r / 1 \) was solved by Morse [4] at \( r = 2 \) and white et al. [6] Who obtained the solution in the form of a generating function and the probabilities could be obtained by a power series expansion. Ritha and Sreelekha[5] treated Fuzzy N-Policy queues with infinite capacity. Al Seedy [1] gave an analytical solution of the queue: \( M / E_r / 1 / N \) with balking only. This work had been followed by Kotb [3] who studied the analytical solution of the state-dependent Erlangian queue: \( M / E_r / 1 / N \) with balking by using a very useful lemma. El- paoumy[2] studied the same system without retention reneged and fuzzy concepts.

In this paper we treat the analytical solution of the queue: \( FM / FE_r / 1 / N (\zeta, \beta) \) for finite capacity considering by using a recurrence relations. We obtain \( \bar{P}_{n,s} \), the probabilities that there are \( n \) units in the system and the unit in service occupies stage \( s \) \( (1 \leq s \leq r) \) in terms of \( \bar{P}_0 \). We consider retention of reneged customers that is, the reneged customer may leave the queue without getting service with probability \( p \) and may remain in the queue for his service with probability \( q = 1 - p \).

The probability of an empty system \( P_0 \) is also obtained. The discipline considered is first in first out (FIFO).

2- THE PROBLEM ANALYSIS

Consider the single – channel service time Erlangian queue having \( r \) – service stages each with rate \( \mu_n \), with the state – dependent. The mean service rate is given by:

\[
\mu_n = \begin{cases} 
\begin{align*}
  r \mu_1 & , \quad n = 1, \\
  r \mu_1 + (n - 1)\zeta p & , \quad 2 \leq n \leq k , \quad \mu_1 < \mu_2 , \\
  r \mu_2 + (n - 1)\zeta p & , \quad k + 1 \leq n \leq N ,
\end{align*}
\end{cases}
\]

where \( \zeta \) is the rate of time \( t \), having the \( (P.d.f) \) given by:
\[ f(t) = \zeta e^{-\zeta t}, \quad t \geq 0, \quad \zeta > 0. \]

This means that the units are served with two different rates \( r\mu_i \) or \( r\mu_2 \) depending on the number of units in the system whether \( 1 \leq n \leq k \) or \( k + 1 \leq n \leq N \) respectively.

Also, consider an exponential inter-arrival pattern with rate \( \lambda_n \). Assume \( (1 - \beta) \) be the probability that a unit balks (does not enter the queue).

where: \( \beta = \text{pr. \{a unit joins the queue\}} \), \( 0 \leq \beta < 1 \), \( 1 \leq n \leq N \);

For \( \beta = 1 \), \( n = 0 \), it is clear that:

\[
\lambda_n = \begin{cases} 
\lambda, & n = 0 \\
\beta \lambda, & 1 \leq n < N \\
0, & n = N
\end{cases}
\]

Assume the probabilities:

\[ P_{n,s} = \text{pr. \{n units in the system and the unit in service being in stage s\}}, \]

where: \( 1 \leq n \leq N \), \( 1 \leq s \leq r \).

\( P_0 \) = probability of an empty system, i.e. the daily probability.

The steady – state difference equations are:

\[ \lambda P_0 - r\mu_1 P_{1,1} = 0, \quad n = 0 \]  \hspace{1cm} (1)

\[ (r\mu_i + \beta\lambda)P_{s,i} - r\mu_i P_{s+1,i} = 0, \quad s = 1 \leq s \leq r - 1 \]

\[ (r\mu_i + \beta\lambda)P_{s,i} - r\mu_i P_{s+1,i} = 0, \quad s = 1 \leq s \leq r - 1 \]  \hspace{1cm} (2)

\[ (r\mu_i + (n - 1)\zeta p + \beta\lambda)P_{s,i} - (r\mu_i + (n - 1)\zeta p)P_{s+1,i} = 0, \quad 1 \leq s \leq r - 1 \]

\[ (r\mu_i + (n - 1)\zeta p + \beta\lambda)P_{s,i} - (r\mu_i + (n - 1)\zeta p)P_{s+1,i} = 0, \quad s = r \]  \hspace{1cm} (3)

\[ (r\mu_i + (k - 1)\zeta p + \beta\lambda)P_{k,i} - \beta\lambda P_{k-1,i} - (r\mu_i + (k - 1)\zeta p)P_{k+1,i} = 0, \quad 1 \leq s \leq r - 1 \]

\[ (r\mu_i + (k - 1)\zeta p + \beta\lambda)P_{k,i} - \beta\lambda P_{k-1,i} - (r\mu_i + (k - 1)\zeta p)P_{k+1,i} = 0, \quad s = r \]  \hspace{1cm} (4)

\[ (r\mu_2 + (n - 1)\zeta p + \beta\lambda)P_{s,i} - \beta\lambda P_{s-1,i} - (r\mu_2 + (n - 1)\zeta p)P_{s+1,i} = 0, \quad 1 \leq s \leq r - 1 \]

\[ (r\mu_2 + (n - 1)\zeta p + \beta\lambda)P_{s,i} - \beta\lambda P_{s-1,i} - (r\mu_2 + (n - 1)\zeta p)P_{s+1,i} = 0, \quad s = r \]  \hspace{1cm} (5)

\[ (r\mu_2 + (N - 1)\zeta p)P_{s,i} - \beta\lambda P_{s-1,i} - (r\mu_2 + (N - 1)\zeta p)P_{s+1,i} = 0, \quad 1 \leq s \leq r - 1 \]

\[ (r\mu_2 + (N - 1)\zeta p)P_{s,i} - \beta\lambda P_{s-1,i} = 0, \quad s = r \]  \hspace{1cm} (6)

Summing (2) over s and using (1), gives

\[ P_{2,1} = \frac{\beta\lambda}{(r\mu_i + \zeta p)} \sum_{s=1}^{r} P_{s,1}, \quad n = 2 \]  \hspace{1cm} (7)
Summing (3) over s, using (7) and adding the results obtaining for \( 2 \leq n \leq k - 1 \), leads to:

\[
P_{n,j} = \frac{\beta \lambda}{r \mu_i + (n-1) \xi p} \sum_{s=1}^{j} P_{n-1,s}, \quad 3 \leq n \leq k
\]

(8)

Similarly, summing (4) over s, and using (8) at n=k, yields

\[
P_{k+1,j} = \frac{\beta \lambda}{r \mu_k + k \xi p} \sum_{s=1}^{j} P_{k,s}, \quad n = k + 1
\]

(9)

Summing (5) over s, and using (9):

\[
P_{n,1} = \frac{\beta \lambda}{r \mu_2 + (n-1) \xi p} \sum_{s=1}^{r} P_{n-1,s}, \quad k + 2 \leq n \leq N
\]

(10)

From equation one can easily show that

\[
P_{r,1} = \varphi_1 P_0
\]

Making use of equation (2), yields

\[
P_{r,s} = \varphi_1 (1 + \beta \varphi_1)^{r-s} P_0, \quad 1 \leq s \leq r
\]

(11)

Upon using the first equation of (3) and (8) we get the recurrence relation.

\[
P_{n,j} = \beta \varphi_n (1 + \beta \varphi_n)^{j-1} \left\{ \sum_{i=1}^{j} P_{n-1,i} - \sum_{i=1}^{j-1} \left( \frac{1}{1 + \beta \varphi_n} \right)^i P_{n-1,i} \right\}, \quad 2 \leq n \leq k
\]

(12)

Also, from the first Equation of (5) and (10), we obtain

\[
P_{n,j} = \beta \varphi_n (1 + \beta \varphi_n)^{j-1} \left\{ \sum_{i=1}^{j} P_{n-1,i} - \sum_{i=1}^{j-1} \left( \frac{1}{1 + \beta \varphi_n} \right)^i P_{n-1,i} \right\}, \quad n = k + 1 \leq n \leq N - 1
\]

(13)

Finally, using equation (6) and equation (10) at n=N, gives:

\[
P_{N,j} = \beta \varphi_N \sum_{i=1}^{r} P_{N-1,i}, \quad 1 \leq s \leq r
\]

(14)

where:

\[
\varphi_n = \begin{cases} 
\frac{\lambda}{r \mu_i + (n-1) \xi p}, & 1 \leq n \leq k \\
\frac{\lambda}{r \mu_2 + (n-1) \xi p}, & k + 1 \leq n \leq N 
\end{cases}
\]

Equations (11) – (14) are the required recurrence relations, that give all probabilities in terms of \( P_0 \) which it-self may now be determined by using the normalizing condition:

\[
P_0 + \sum_{n=1}^{N} \sum_{s=1}^{r} P_{n,s} = 1
\]

(15)

Hence all the probabilities are completely known in terms of the queue parameters.
3- Example

The following example illustrates the theoretical results. In the system: $M/E_r/1/N$ with state-dependent balking, reneging and retention reneged, let $k = 2$, $r = 3$ and $N = 4$, (i.e., the queue $M/E_3/1/4(\zeta, \beta)$), in the equations (11) – (15), the results are:

\[
P_{1,1} = a_1 P_0, \quad P_{1,2} = a_2 P_0, \quad P_{1,3} = a_3 P_0,
\]
\[
P_{2,1} = b_1 P_0, \quad P_{2,2} = b_2 P_0, \quad P_{2,3} = b_3 P_0,
\]
\[
P_{3,1} = c_1 P_0, \quad P_{3,2} = c_2 P_0, \quad P_{3,3} = c_3 P_0,
\]
\[
P_{4,1} = d_1 P_0, \quad P_{4,2} = d_2 P_0, \quad P_{4,3} = d_3 P_0.
\]

where:

\[
a_1 = \varphi_1, \quad a_2 = \varphi_1 (1 + \beta \varphi_1), \quad a_3 = \varphi_1 (1 + \beta \varphi_1)^2,
\]
\[
b_1 = \beta \varphi_2 (a_1 + a_2 + a_3),
\]
\[
b_2 = \beta \varphi_2 (1 + \beta \varphi_2) \left\{ a_1 + a_2 + a_3 - \left( \frac{1}{1 + \beta \varphi_2} \right) a_1 \right\},
\]
\[
b_3 = \beta \varphi_2 (1 + \beta \varphi_2) \left\{ a_1 + a_2 + a_3 - \left( \frac{1}{1 + \beta \varphi_2} \right) a_1 - \left( \frac{1}{1 + \beta \varphi_2} \right)^2 a_2 \right\},
\]
\[
c_1 = \beta \varphi_3 (b_1 + b_2 + b_3),
\]
\[
c_2 = \beta \varphi_3 (1 + \beta \varphi_3) \left\{ b_1 + b_2 + b_3 - \left( \frac{1}{1 + \beta \varphi_3} \right) b_1 \right\},
\]
\[
c_3 = \beta \varphi_3 (1 + \beta \varphi_3) \left\{ b_1 + b_2 + b_3 - \left( \frac{1}{1 + \beta \varphi_3} \right) b_1 - \left( \frac{1}{1 + \beta \varphi_3} \right)^2 b_2 \right\},
\]
\[
d_1 = \beta \varphi_4 (c_1 + c_2 + c_3), \quad d_2 = \beta \varphi_4 (c_2 + c_3), \quad d_3 = \beta \varphi_4 c_3,
\]
\[
\varphi_1 = \frac{\lambda}{3 \mu_1}, \quad \varphi_2 = \frac{\lambda}{3 \mu_1 + \zeta p}, \quad \varphi_3 = \frac{\lambda}{3 \mu_2 + 2 \zeta p}, \quad \varphi_4 = \frac{\lambda}{3 \mu_2 + 3 \zeta p}.
\]

From the normalizing condition:

\[
P_0 + \sum_{s=1}^3 P_{1,s} + \sum_{s=1}^3 P_{2,s} + \sum_{s=1}^3 P_{3,s} + \sum_{s=1}^3 P_{4,s} = 1,
\]

we have:

\[
P_0 = \left\{ 1 + a_1 + a_2 + a_3 + b_1 + b_2 + b_3 + c_1 + c_2 + c_3 + c_1 + d_2 + d_3 \right\}^{-1}.
\]

Therefore, the expected numbers in the system and in the queue are, respectively.
\[ L = \sum_{n=1}^{4} \sum_{s=1}^{3} n P_{n,s} = \left\{ a_1 + a_2 + a_3 + 2(b_1 + b_2 + b_3) + 3(c_1 + c_2 + c_3) + 4(d_1 + d_2 + d_3) \right\} P_0 \]

\[ L_q = \sum_{n=1}^{4} \sum_{s=1}^{3} (n-1) P_{n,s} = \left\{ (b_1 + b_2 + b_3) + 2(c_1 + c_2 + c_3) + 3(d_1 + d_2 + d_3) \right\} P_0 \]

Also the expected waiting time in Kotb the system and the queue are obtained as follows:

\[ W = \frac{L}{\lambda_{eff}}, \quad W_q = \frac{L_q}{\lambda_{eff}}, \quad \lambda_{eff} = (L - L_q) \mu, \quad \text{and} \quad \mu = \frac{1}{2}(\mu_1 + \mu_2) \]

where \( \lambda_{eff} \) is the mean rate of units actually entering the system.

4. FUZZY SYSTEM

We consider an extension of reneging of customers queuing model with finite capacity in which arriving customers follow a Poisson process with a fuzzy arrival rate \( \tilde{\lambda} \) and service times are Erlangian with a fuzzy service rate \( \tilde{\mu}_n \). \( \tilde{\lambda} \) and \( \tilde{\mu}_n \) are imprecise and uncertain.

If \( \tilde{\lambda}, \tilde{\mu}_1 \) and \( \tilde{\mu}_2 \) are defined by triangular fuzzy numbers such that :

\[ \tilde{\lambda} = [\lambda_1, \lambda_2, \lambda_3]; \quad \tilde{\mu}_1 = [\mu_{11}, \mu_{12}, \mu_{13}], \quad \tilde{\mu}_2 = [\mu_{21}, \mu_{22}, \mu_{23}] \quad \text{and} \quad \tilde{\mu} = [\mu_1, \mu_2, \mu_3] \]

Where: \( \lambda_1 < \lambda_2 < \lambda_3 \), \( \mu_{11} < \mu_{12} < \mu_{13} \), \( \mu_{21} < \mu_{22} < \mu_{23} \) and \( \mu_1 < \mu_2 < \mu_3 \).

The membership function of \( \eta_\lambda(\tilde{\lambda}) \) and \( \eta_\mu(\tilde{\mu}) \) are defined as follows.

\[
\eta_\lambda(\tilde{\lambda}) = \begin{cases} 
0, & \text{if } \lambda < \lambda_1 \\
\frac{\lambda - \lambda_1}{\lambda_2 - \lambda_1}, & \text{if } \lambda_1 \leq \lambda \leq \lambda_2 \\
\frac{\lambda_3 - \lambda}{\lambda_3 - \lambda_2}, & \text{if } \lambda_2 \leq \lambda \leq \lambda_3 \\
0, & \text{if } \lambda \geq \lambda_3 
\end{cases}
\]

\[
\eta_\mu(\tilde{\mu}) = \begin{cases} 
0, & \text{if } \mu < \mu_1 \\
\frac{\mu - \mu_1}{\mu_2 - \mu_1}, & \text{if } \mu_1 \leq \mu \leq \mu_2 \\
\frac{\mu_3 - \mu}{\mu_3 - \mu_2}, & \text{if } \mu_2 \leq \mu \leq \mu_3 \\
0, & \text{if } \mu \geq \mu_3 
\end{cases}
\]

Using the concept of \( \alpha - \)cut method and the operations on triangular fuzzy numbers to find fuzzy probabilities and fuzzy performance measures.

\[
\alpha = [\alpha(\lambda_2 - \lambda_1) + \lambda_1, \lambda_3 - \alpha(\lambda_3 - \lambda_2)]; \quad \alpha = [\alpha(\mu_2 - \mu_1) + \mu_1, \mu_3 - \alpha(\mu_3 - \mu_2)]
\]

\[
a[\tilde{\phi}_1] = \left[ \frac{\alpha(\lambda_2 - \lambda_1) + \lambda_1}{3(\mu_{13} - \alpha(\mu_{13} - \mu_{12}))}, \frac{\lambda_1 - \alpha(\lambda_3 - \lambda_2)}{3(\mu_{13} - \alpha(\mu_{13} - \mu_{12})) + 3(\alpha(\mu_{12} - \mu_{11}) + \mu_{11})} \right],
\]

\[
a[\tilde{\phi}_2] = \left[ \frac{\alpha(\lambda_2 - \lambda_1) + \lambda_1}{3(\mu_{13} - \alpha(\mu_{13} - \mu_{12})) + 3(\alpha(\mu_{12} - \mu_{11} + \mu_{11}) + \zeta \cdot p)}, \frac{\lambda_1 - \alpha(\lambda_3 - \lambda_2)}{3(\mu_{13} - \alpha(\mu_{13} - \mu_{12})) + 3(\alpha(\mu_{12} - \mu_{11} + \mu_{11}) + \zeta \cdot p)} \right],
\]

\[
a[\tilde{\phi}_3] = \left[ \frac{\alpha(\lambda_2 - \lambda_1) + \lambda_1}{3(\mu_{23} - \alpha(\mu_{23} - \mu_{22})) + 2\zeta \cdot p), \frac{\lambda_1 - \alpha(\lambda_3 - \lambda_2)}{3(\mu_{23} - \alpha(\mu_{23} - \mu_{22})) + 3(\alpha(\mu_{22} - \mu_{21}) + \mu_{21}) + 2\zeta \cdot p)} \right],
\]

\[
a[\tilde{\phi}_4] = \left[ \frac{\alpha(\lambda_2 - \lambda_1) + \lambda_1}{3(\mu_{23} - \alpha(\mu_{23} - \mu_{22})) + 3(\alpha(\mu_{22} - \mu_{21}) + \mu_{21}) + 3\zeta \cdot p)}, \frac{\lambda_1 - \alpha(\lambda_3 - \lambda_2)}{3(\mu_{23} - \alpha(\mu_{23} - \mu_{22})) + 3(\alpha(\mu_{22} - \mu_{21}) + \mu_{21}) + 3\zeta \cdot p)} \right],
\]
\[
\tilde{P}_{1,1} = \tilde{a}_1 \tilde{P}_0, \quad \tilde{P}_{1,2} = \tilde{a}_2 \tilde{P}_0, \quad \tilde{P}_{1,3} = \tilde{a}_3 \tilde{P}_0.
\]

\[
\tilde{P}_{2,1} = \tilde{b}_1 \tilde{P}_0, \quad \tilde{P}_{2,2} = \tilde{b}_2 \tilde{P}_0, \quad \tilde{P}_{2,3} = \tilde{b}_3 \tilde{P}_0.
\]

\[
\tilde{P}_{3,1} = \tilde{c}_1 \tilde{P}_0, \quad \tilde{P}_{3,2} = \tilde{c}_2 \tilde{P}_0, \quad \tilde{P}_{3,3} = \tilde{c}_3 \tilde{P}_0.
\]

\[
\tilde{P}_{4,1} = \tilde{d}_1 \tilde{P}_0, \quad \tilde{P}_{4,2} = \tilde{d}_2 \tilde{P}_0, \quad \tilde{P}_{4,3} = \tilde{d}_3 \tilde{P}_0.
\]

\[
\tilde{P}_0 = \left\{ 1 + \tilde{a}_1 + \tilde{a}_2 + \tilde{a}_3 + \tilde{b}_1 + \tilde{b}_2 + \tilde{b}_3 + \tilde{c}_1 + \tilde{c}_2 + \tilde{c}_3 + \tilde{d}_1 + \tilde{d}_2 + \tilde{d}_3 \right\}^{-1}.
\]

\[
a[\mathbf{L}] = \frac{\tilde{a}_1 + \tilde{a}_2 + \tilde{a}_3 + 2(\tilde{b}_1 + \tilde{b}_2 + \tilde{b}_3) + 3(\tilde{c}_1 + \tilde{c}_2 + \tilde{c}_3) + 4(\tilde{d}_1 + \tilde{d}_2 + \tilde{d}_3)}{1 + \tilde{a}_1 + \tilde{a}_2 + \tilde{a}_3 + \tilde{b}_1 + \tilde{b}_2 + \tilde{b}_3 + \tilde{c}_1 + \tilde{c}_2 + \tilde{c}_3 + \tilde{d}_1 + \tilde{d}_2 + \tilde{d}_3}.
\]

\[
\alpha[\mathbf{L}] = \left[ L^l_a, L^u_a \right].
\]

\[
a[\mathbf{L}_q] = \frac{\tilde{b}_1 + \tilde{b}_2 + \tilde{b}_3 + 2(\tilde{c}_1 + \tilde{c}_2 + \tilde{c}_3) + 3(\tilde{d}_1 + \tilde{d}_2 + \tilde{d}_3)}{1 + \tilde{a}_1 + \tilde{a}_2 + \tilde{a}_3 + \tilde{b}_1 + \tilde{b}_2 + \tilde{b}_3 + \tilde{c}_1 + \tilde{c}_2 + \tilde{c}_3 + \tilde{d}_1 + \tilde{d}_2 + \tilde{d}_3}.
\]

\[
\alpha[\mathbf{L}_q] = \left[ L^l_{aq}, L^u_{aq} \right].
\]

Where: \( L^l_a = \frac{a_{11} + a_{21} + a_{31} + 2(b_{11} + b_{21} + b_{31}) + 3(c_{11} + c_{21} + c_{31}) + 4(d_{11} + d_{21} + d_{31})}{1 + a_{11} + a_{21} + a_{31} + b_{11} + b_{21} + b_{31} + c_{11} + c_{21} + c_{31} + d_{11} + d_{21} + d_{31}} \),

\[
L^u_a = \frac{a_{12} + a_{22} + a_{32} + 2(b_{12} + b_{22} + b_{32}) + 3(c_{12} + c_{22} + c_{32}) + 4(d_{12} + d_{22} + d_{32})}{1 + a_{12} + a_{22} + a_{32} + b_{12} + b_{22} + b_{32} + c_{12} + c_{22} + c_{32} + d_{12} + d_{22} + d_{32}}.
\]

\[
\bar{L} = \left[ L^l_{aq} \bigg|_{a=0}, L^l_a \bigg|_{a=1}, L^u_a \bigg|_{a=0} \right].
\]

\[
\bar{L}^l_{aq} = \frac{b_{1} + b_{2} + b_{3} + 2(c_{1} + c_{2} + c_{3}) + 3(d_{1} + d_{2} + d_{3})}{1 + a_{1} + a_{2} + a_{3} + b_{1} + b_{2} + b_{3} + c_{1} + c_{2} + c_{3} + d_{1} + d_{2} + d_{3}}.
\]

\[
\bar{L}^u_{aq} = \frac{b_{1} + b_{2} + b_{3} + 2(c_{1} + c_{2} + c_{3}) + 3(d_{1} + d_{2} + d_{3})}{1 + a_{1} + a_{2} + a_{3} + b_{1} + b_{2} + b_{3} + c_{1} + c_{2} + c_{3} + d_{1} + d_{2} + d_{3}}.
\]

And: \( L^l_{aq} = \frac{b_{1} + b_{2} + b_{3} + 2(c_{1} + c_{2} + c_{3}) + 3(d_{1} + d_{2} + d_{3})}{1 + a_{1} + a_{2} + a_{3} + b_{1} + b_{2} + b_{3} + c_{1} + c_{2} + c_{3} + d_{1} + d_{2} + d_{3}} \).

Also, as before we find:

\[
\bar{L}_q = \left[ L_{q1}, L_{q2}, L_{q3} \right]
\]

Where:

\[
L_{q1} = L^l_{aq} \bigg|_{a=0}, \quad L_{q2} = L^l_{aq} \bigg|_{a=1}, \quad \text{and} \quad L_{q3} = L^u_{aq} \bigg|_{a=0}
\]

\[
L^l_{aq} = \frac{b_{1} + b_{2} + b_{3} + 2(c_{1} + c_{2} + c_{3}) + 3(d_{1} + d_{2} + d_{3})}{1 + a_{1} + a_{2} + a_{3} + b_{1} + b_{2} + b_{3} + c_{1} + c_{2} + c_{3} + d_{1} + d_{2} + d_{3}}.
\]

\[
L^u_{aq} = \frac{b_{1} + b_{2} + b_{3} + 2(c_{1} + c_{2} + c_{3}) + 3(d_{1} + d_{2} + d_{3})}{1 + a_{1} + a_{2} + a_{3} + b_{1} + b_{2} + b_{3} + c_{1} + c_{2} + c_{3} + d_{1} + d_{2} + d_{3}}.
\]
From Fuzzy Little's formula, we get:

\[ \bar{W} = \frac{L}{\lambda}, \quad \bar{W}_q = \frac{L_q}{\lambda_{\text{eff}}} \quad \text{and} \quad \tilde{\lambda}_{\text{eff}} = (\bar{L} - \bar{L}_q) \tilde{\mu}. \]

5. **SPECIAL CASES**

**Case 1:** Let \( \zeta = 0 \) and \( k \to \infty \left( \mu_1 = \mu_2 = \mu, \ k = N \right) \). without fuzzy concepts.

Our results agree with the results of Al-Seedy [1].

**Case 2:** Results of both [3] and [1], has been obtained by letting \( \zeta = 0 \) in the equations (11) – (14) in our results.

**Case 3:** Results of [6], can be obtained by letting \( k, N \to \infty, \zeta = 0 \) and \( \beta = 1 \) in our results.

**Case 4** When there is no customer retention \( q = 0 \) and \( p = 1 \). Therefore, the queuing model without fuzzy concept studied in this paper reduces to \( M/\mu/E_1/1/N \) with state-dependent service rate, balking and reneging as in [2].

6. **CONCLUSION**

In this paper, the truncated Erlangian service queue is studied with state-dependent, balking, reneging, retention of reneged customers and fuzzy parameter. The recurrence relations that gave all the probabilities in terms of \( P_0 \) are derived. We illustrate the method by an example is give to obtain some performance measures such as \( \bar{L} \) and \( \bar{L}_q \).

**REFERENCES**