Multiple Degree Reduction of Interval DP Curves

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Abstract—Fixed DP curve is a recent representation of the polynomial curves, proposed by Delgado and Pena in 2003. They introduced a new curve representation with linear computational complexity. An algorithmic approach to degree reduction of interval DP curve is presented in this paper. The four fixed Kharitonov's polynomials (four fixed DP curves) associated with the original interval DP curve are obtained. These four fixed DP curves are transformed into four fixed Bezier curves. The degree of the four fixed Bezier curves is reduced based on the matrix representations of the degree reduction process. The process of degree reduction k times are applied to the four fixed Bezier curves of degree n to obtain the four fixed Bezier curves of degree n – k without changing their shapes. The four fixed reduced Bezier curves are converted into DP curves of the same degree. Finally the reduced interval DP control points are obtained from the four fixed reduced DP control points. An illustrative example is included in order to demonstrate the effectiveness of the proposed method.

Index Term—Image processing, CAGD, degree reduction, interval DP curve, interval Bezier curve.

I. INTRODUCTION

In 2003, Delgado and Pena [1], [2] introduced a new parametric curve representation of which properties are normalized totally positive (NTP) basis functions, corner cutting algorithm and linear computational complexity. Previously, it was called DP-Ball curve [3], [4]. However, it has no relation with any of generalized Ball curves. Thus, it has been called DP curve (or Delgado-Pena curve in this paper). Parametric polynomial curves have some important properties that make them well suited for geometric modeling.

Classically, the presentation of Bezier curves [5], [6], [7], [8] has been widely used in many CAD/CAM systems because they present the simplest model, defined in terms of the Bernstein basis polynomials. Among several important properties of Bezier curves are convexity, affine invariance, and Bernstein polynomial symmetry. The degree elevation and reduction [9] of Bezier curve are two promising approaches that can be proficiently applied into many consequent properties.

In image processing and visualization, comparing two bitmapped images needs to be compared from their pixels by matching pixel-by-pixel. Consequently, it takes a lot of computational time while the comparison of two vector-based images is significantly faster. Sometimes these raster graphics images can be approximately converted into the vector-based images by various techniques. After comparison, the problem of comparing two raster graphics images can be reduced to the problem of comparing vector graphics images. Hence, the problem of comparing pixel-by-pixel can be reduced to the problem of polynomial comparisons. In computer aided geometric design (CAGD), the vector graphics images are the composition of curves and surfaces. Curves are defined by a sequence of control points and their polynomials.

In internet applications, there are many popular search engines such as Google, Yahoo and MSN. These search engines have provided efficient mechanisms in searching for the relevant media or documents from a given set of keywords. Nevertheless, textual comparisons can only be accomplished. Although some pictures can be matched to the given keywords, the comparisons are attained by matching from the information provided in those pictures. However, there are several multimedia depicted in the forms of pictures, figures or images. Thus, it is interesting to introduce the algorithms for seeking for these kinds of information.

Typically, images can be classified into two categories: raster graphics and vector graphics images. A raster graphics image is represented by a rectangular grid of pixels whereas vector graphics image is defined by a set of mathematical equations representing the geometric objects, e.g., points, lines, polygons, curves, and surfaces.

The information contained in the raster graphics image is a collection of pixel attributes: the coordinates and colors. Comparing two images, one needs to compare pixel-by-pixel, coordinate-by-coordinate or even color-by-color. Consequently, comparing two large images, e.g., photographic image in the Internet or in the image banks, it is inevitably needed to compare a plenty of images. Moreover, seeking for a simple geometric object composed in a bitmap image is concerned as a complicated task. In vector-based images, each element is represented in terms of the mathematic formula and its attributes. There are many properties of those geometric primitives that can make the image comparison easier. Using the relevant properties instead of computing the whole image can reduce the computational time. Fortunately in some particular applications, raster graphics images can be converted into the vector-based images. It is reasonable to transform raster graphics images into vector graphics images and compares those vector graphics images. In computer aided geometric design (CAGD), a vector graphics image is an aggregation of curves and surfaces. Curves and surfaces can be modeled in various techniques. One of those methods that has been commonly used is the polynomial curve and surface representation. There are several kinds of polynomial curves in CAGD, e.g., Bezier [5], [6], [7], [8] Said-Ball [10], Wang-Ball [11], [12], [13], B-Spline curves [14] and DP curves [1], [2]. These curves have some common and different properties. All of them are defined in terms of the sum of product of their blending functions and control points. They are just different in their own basis polynomials. In order to compare these curves, we need to consider these properties. The common properties of these curves are control points, weights, and their number of degrees. Control points are the points that affect to the shape of the curve. Weights can be treated as the indicators to control how much each control point influences to the curve. Number of degree determines the maximum degree of polynomials. In different curves, these properties are not computed by the same method. To compare different kinds of curves we need to convert these curves into an intermediate form.
A lot of research [15-30] effort has gone into curves and surfaces in the last 30 years because of these reasons. Many sophisticated curve methods are known today-some are specialized and others are general purpose.

In this work, the degree reducing matrix for the four fixed Bezier curves will be used to obtain the degree reduction of the four fixed DP curves associated with the original interval DP curve. First, the relationships between Bezier and DP fixed control points were used for converting DP fixed control points into Bezier fixed control points of degree \( n \). The degree of the four fixed Bezier curves will then be reduced by one using the degree reduction algorithm. Finally, the \( (n - k) \)-degree four fixed Bezier curves can be readily transformed into the four fixed DP curves of degree \( n - k \), and reduced interval DP control points can be obtained from the four fixed reduced DP control points.

This paper is organized in the following sections. Section II describes interval Bezier curves, and section III includes interval DP curves whereas section IV provides conversion from interval DP to interval Bezier control points, while section V presents conversion from interval Bezier into interval DP control points and section VI offers an interval DP degree reduction. Section VII shows a numerical example, and the final section offers conclusions.

II. INTERVAL BEZIER CURVES

Bezier curve provides a simple model of constructing a parametric curve. A parametric equation of Bezier curve can be defined by the linear combination of the Bernstein polynomials and its interval control points. The interval control points can also be used for determining the shape of curve. Although, the curve does not pass through its control points, it passes closely to its control points. Although, the curve does not pass through its control points, it passes closely to its control points. Although, the curve does not pass through its control points, it passes closely to its control points. Although, the curve does not pass through its control points, it passes closely to its control points.

\[
B_k^j(u) = \binom{j}{k}(1 - u)^(j - k)u^k, \quad (k = 0, 1, \ldots, j) \tag{2}
\]

and \( \binom{j}{k} = \frac{j!}{k!(j-k)!} \) is a binomial coefficient.

The four fixed Kharitonov’s polynomials (four fixed Bezier curves) [31] associated with the original interval Bezier curve are:

\[
P_n^4(u) = p_n^+ + p_{n-1}^-u + p_{n-2}^-u^2 + p_{n-3}^-u^3 + p_{n-4}^-u^4 + p_{n-5}^-u^5 + \cdots \\
\equiv a_n^+u + a_{n-1}^+u^2 + \cdots + a_0^+u^n
\]

The degree reduction of fixed Bezier curve [15] is a method used for reducing a vertex from the given interval control points but the shape of the curve remains the same. In this work, interval Bezier degree reduction will be employed as an intermediate step to obtain the required reduced interval DP curve. The vertices of the new four fixed Kharitonov’s polygon (four fixed Bezier polygon), denoted by \( \{a_j^{(n)}\}_{j=0}^{n} \) for \( j = 1,2,3,4 \) can be computed from the original fixed control points, \( \{a_i^n\}_{i=0}^{n} \) by the following proposition:

**Proposition 1.** The degree reduction for the four fixed Bezier curves of degree \( n \) into \( n - 1 \) can be expressed in terms of the matrix representation.

\[
\begin{bmatrix}
a_0^{(n-1)} & a_1^{(n-1)} & \ldots & a_{n-1}^{(n-1)}
\end{bmatrix}
\begin{bmatrix}
\alpha_0^n & \alpha_1^n & \ldots & \alpha_n^n
\end{bmatrix}
\begin{bmatrix}
m_0 \cdot n_0 & m_1 \cdot n_1 & \ldots & m_n \cdot n_n
\end{bmatrix}
\begin{bmatrix}
\alpha_0^{n-1} & \alpha_1^{n-1} & \ldots & \alpha_n^{n-1}
\end{bmatrix}
\right]
\]

which can be created from a square matrix \( C \) as shown in equation (7).

\[
C = \begin{bmatrix}
c_{0,0} & c_{0,1} & \ldots & c_{0,n} \\
c_{1,0} & c_{1,1} & \ldots & c_{1,n} \\
\vdots & \vdots & \ddots & \vdots \\
c_{n-1,0} & c_{n-1,1} & \ldots & c_{n-1,n} \\
c_{n,0} & c_{n,1} & \ldots & c_{n,n}
\end{bmatrix}
\]

where,

\[
c_{i,j} = \begin{cases}
\frac{(n)!}{(i-j)!}, & 0 \leq i \leq n - 1 \text{ and } 0 \leq j \leq n \\
1, & \text{for } i = j \text{ and } n + 1 \leq j \leq n + k \\
0, & \text{otherwise}
\end{cases}
\]

and \( k \) is number of reducing degree reduction (\( k = 1 \)). After inverting a square matrix \( C \) the degree reduction matrix \( M_n^{n-1} \) can be obtained as follows:
The decreasing vertices of Bezier polygon in one process can reduce loops for calculating the new set of vertices of degree. The Bezier degree reduction can be rewritten in term of transformation matrix as in the following proposition.

**Proposition 2.** The four fixed Bezier curves multiple degree reduction of degree into can be written as:

\[
\begin{bmatrix}
\ell_{n-1,0} & \ell_{n-1,1} & \cdots & \ell_{n-1,n} & | & \ell_{n-1,n+1} \\
\ell_{n,0} & \ell_{n,1} & \cdots & \ell_{n,n} & | & \ell_{n,n+1} \\
\vdots & \vdots & \ddots & \vdots & | & \vdots \\
\ell_{n+k,0} & \ell_{n+k,1} & \cdots & \ell_{n+k,n} & | & \ell_{n+k,n+1}
\end{bmatrix}
\]

where \(M^{n-k}_n = (\ell_{11})_{n \times (n+1)}\) is the multiple degree reduction matrix of degree that can be defined by:

\[
M^{n-k}_n = \begin{bmatrix}
m_{0,0} & m_{0,1} & \cdots & m_{0,n} \\
m_{1,0} & m_{1,1} & \cdots & m_{1,n} \\
\vdots & \vdots & \ddots & \vdots \\
m_{n-k,0} & m_{n-k,1} & \cdots & m_{n-k,n}
\end{bmatrix}
\]

Equation (7) is used for creating the square matrix \(C\) (\(k\) is the positive integer), then inverting a square matrix \(C\) and obtaining the multiple degree reduction matrix \(M^{n-k}_n\) as follows:

\[
C^{-1} = \begin{bmatrix}
\ell_{0,0} & \ell_{0,1} & \cdots & \ell_{0,n} & | & \ell_{0,n+1} \\
\ell_{1,0} & \ell_{1,1} & \cdots & \ell_{1,n} & | & \ell_{1,n+1} \\
\vdots & \vdots & \ddots & \vdots & | & \vdots \\
\ell_{n-k,0} & \ell_{n-k,1} & \cdots & \ell_{n-k,n} & | & \ell_{n-k,n+1} \\
\ell_{n-k,0} & \ell_{n-k,1} & \cdots & \ell_{n-k,n} & | & \ell_{n-k,n+1} \\
\ell_{n-k,0} & \ell_{n-k,1} & \cdots & \ell_{n-k,n} & | & \ell_{n-k,n+1} \\
\vdots & \vdots & \ddots & \vdots & | & \vdots \\
\ell_{n+k,0} & \ell_{n+k,1} & \cdots & \ell_{n+k,n} & | & \ell_{n+k,n+1}
\end{bmatrix}
\]

and

\[
C^{-1} = \begin{bmatrix}
(\ell_{11})_{n \times (n+1)} & | & (\ell_{12})_{n \times (n+1)} \\
(\ell_{21})_{2 \times (n+1)} & | & (\ell_{22})_{2 \times (n+1)}
\end{bmatrix}
\]

and the multiple degree reduction matrix will be:

\[
M^{n-k}_n = (\ell_{11})_{(n-k+1) \times (n+1)}
\]

**III. INTERVAL DP CURVES**

There are three parametric curves, e.g., Bezier, Said-Ball and DP curves, proven to be totally positive. Among these curves, Bezier curve has been known to be the best approximation of the control polygon because the curve is closest to its control polygon. This property indicates to the efficiency of evaluating the points on DP curves. In this work we introduced an interval DP curves, which can be defined as follows:

Let \([d_i^- , d_i^+]\) be a given set of interval control points which defines the interval DP curve:

\[
D^n_i (u) = \sum_{i=0}^{n} [d_i^- , d_i^+] D^n_i (u), \quad 0 \leq u \leq 1
\]

of degree \(n\) where \(D^n_i (u)\) are the DP blending functions defined as following:

\[
D^n_i (u) = \begin{cases}
(1 - u)^n, & \text{for } i = 0 \\
(1 - u)^i u^n, & \text{for } 1 \leq i \leq \frac{n}{2} - 1 \\
(1 - u)^{\frac{n}{2}}, & \text{for } i = \frac{n}{2} \\
K^n_i (u) + K^n_{i+1}, & \text{for } i = \frac{n}{2} \\
K^n_{i+1}(1 - u) & \text{for } \frac{n}{2} + 1 \leq i \leq n
\end{cases}
\]

where,

\[
K^n_i (u) = \left(\frac{u}{\frac{n}{2}}\right)^i u^{\frac{n}{2} - i} - \left(1 - u\right)^{\frac{n}{2} - i}
\]

The four fixed Kharitonov's polynomials (four fixed DP curves) [31] associated with the original interval DP curve are:

\[
Q^n_1 = q_0^- + q_1^- u + q_2^- u^2 + q_3^- u^3 + q_4^- u^4 + q_5^- u^5 + \cdots \\
\equiv \beta_1^0 + \beta_2^1 u + \beta_3^2 u^2 + \cdots + \beta_n^u u^n
\]

\[
Q^n_2 = q_0^+ + q_1^- u + q_2^+ u^2 + q_3^- u^3 + q_4^- u^4 + q_5^+ u^5 + \cdots \\
\equiv \beta_1^0 + \beta_2^1 u + \beta_3^2 u^2 + \cdots + \beta_n^u u^n
\]

\[
Q^n_3 = q_0^- + q_1^+ u + q_2^- u^2 + q_3^+ u^3 + q_4^+ u^4 + q_5^- u^5 + \cdots \\
\equiv \beta_1^0 + \beta_2^1 u + \beta_3^2 u^2 + \cdots + \beta_n^u u^n
\]

\[
Q^n_4 = q_0^+ + q_1^- u + q_2^+ u^2 + q_3^- u^3 + q_4^- u^4 + q_5^- u^5 + \cdots \\
\equiv \beta_1^0 + \beta_2^1 u + \beta_3^2 u^2 + \cdots + \beta_n^u u^n
\]
The formula for the conversions between fixed DP and fixed Bezier control points, and vice versa is defined and proved in [32] using the polar form approach. But the conversions from interval DP into interval Bezier and from the interval Bezier into DP interval control points, are explicitly shown in the following sections.

IV. CONVERSION FROM INTERVAL DP INTO INTERVAL BEZIER CONTROL POINTS

The conversion from interval DP into interval Bezier control points is provided in terms of the transformation matrix as shown in the following proposition.

**Proposition 3.** The fixed Bezier control points (associated with the four fixed Bezier curves of degree \(n\)) of a corresponding four fixed DP curves of degree \(n\) can be given in terms of the multiplication of the fixed DP control points (for each of the four fixed DP control points associated with the original interval DP curve) and \((n + 1) \times (n + 1)\) conversion matrix \(F_n\), by:

\[
\begin{bmatrix}
\alpha^l_0 & \alpha^l_1 & \ldots & \alpha^l_n
\end{bmatrix} = \begin{bmatrix}
\beta^l_0 & \beta^l_1 & \ldots & \beta^l_n
\end{bmatrix} \cdot [F_n]^T \quad (j = 1, 2, 3, 4)
\]

where \(F_n\) is the conversion matrix from fixed DP control points associated with the four fixed DP curves into the corresponding fixed Bezier control points associated with the four fixed Bezier curves that can be defined by:

\[
F_n = \begin{bmatrix}
f_{0,0} & f_{0,1} & \cdots & f_{0,n} \\
f_{1,0} & f_{1,1} & \cdots & f_{1,n} \\
\vdots & \vdots & \ddots & \vdots \\
f_{n,0} & f_{n,1} & \cdots & f_{n,n}
\end{bmatrix}
\]

and

\[
f_{k,l} = \begin{cases} 
1, & \text{for } k = l = 0 \text{ or } k = l = n \\
\frac{n(n-1)}{l(n-k)}, & \text{for } 1 \leq k \leq \frac{n}{2} - 1 \text{ and } 1 \leq l \leq k \\
\frac{(n-1)(n-l)}{(n-k)(n-l-k)}, & \text{for } \frac{n}{2} + 1 \leq k \leq n-1 \text{ and } k \leq l \leq n-1
\end{cases}
\]

If \(n\) is even then,

\[
f_{k,l} = \begin{cases} 
1 - \frac{\left(\frac{n-l}{2}\right) + 1}{\left(\frac{n}{2}\right) + 1}, & \text{for } k = \frac{n}{2} \text{ and } 1 \leq l \leq \frac{n}{2} \\
1 - \frac{\left(\frac{n-l}{2}\right) + 1}{\left(\frac{n}{2}\right) + 1}, & \text{for } k = \frac{n}{2} \text{ and } \frac{n}{2} \leq l \leq n-1
\end{cases}
\]

If \(n\) is odd then,

\[
f_{k,l} = \begin{cases} 
1 - \frac{\left(\frac{n-l}{2}\right) + 1}{\left(\frac{n}{2}\right) + 1}, & \text{for } k = \frac{n}{2} \text{ and } 1 \leq l \leq \frac{n}{2} \\
1 - \frac{\left(\frac{l}{2}\right) + 1}{\left(\frac{n}{2}\right) + 1}, & \text{for } k = \frac{n}{2} \text{ and } \frac{n}{2} \leq l \leq n-1
\end{cases}
\]

\[
\frac{2(n-l)}{\left(\frac{n}{2}\right) + 1}.
\]

V. CONVERSION FROM INTERVAL BEZIER INTO INTERVAL DP CONTROL POINTS

The conversion from interval Bezier into interval DP control points of the same curve can be rewritten from the equation of the conversion from interval DP into interval Bezier control points. Thus, the relationship can be shown in the following proposition.

**Proposition 4.** The fixed DP control points (associated with the four fixed DP curves of degree \(n\)) of a corresponding four fixed Bezier curves of degree \(n\) can be defined in terms of the multiplication of the fixed Bezier control points (for each of the four fixed Bezier control points associated with the original interval Bezier curve) and the inverse of the \((n + 1) \times (n + 1)\) conversion matrix \(F_n\), by:

\[
\begin{bmatrix}
\beta^l_0 & \beta^l_1 & \ldots & \beta^l_n
\end{bmatrix} = \begin{bmatrix}
\alpha^l_0 & \alpha^l_1 & \ldots & \alpha^l_n
\end{bmatrix} \cdot \left([F_n]^T\right)^{-1} \quad (j = 1, 2, 3, 4)
\]

where \(\left([F_n]^T\right)^{-1}\) is the conversion matrix from fixed Bezier control points associated with the four fixed Bezier curves into the corresponding fixed DP control points associated with the four fixed DP curves, and \(\left([F_n]^T\right)^{-1}\) is the conversion matrix from fixed DP into corresponding fixed Bezier control points, which can be defined exactly the same as in equation (22).

VI. INTERVAL DP DEGREE REDUCTION

In order to find the resulting matrix of the interval DP degree reduction, some of the intermediate steps of the transformations are needed to be computed. For interval DP degree reduction, the Bezier degree reduction matrix associated with the four fixed Bezier curves has to be used in finding the degree reduction of interval DP curve. Interval DP degree reduction can be created by the following steps:
Algorithm for the interval DP Degree Reduction

1. Transform the DP control points associated with the four fixed DP curves of degree $n$ into the Bezier control points of the same curve equation (21).
2. Reduce degree of the four fixed Bezier curves equation (11).
3. Convert the four fixed Bezier control points of degree $n - k$ into the fixed DP control points of the same degree equation (26).
4. The reduced interval DP control points can be obtained from the four fixed reduced DP control points as follows:
   \[ [\beta_i^*, \beta_j^*] = \min(\beta_i^j), \max(\beta_i^j) \]
   \[ (i = 0, 1, \cdots, n - k) \text{ and } (j = 1, 2, 3, 4) \]

   Thus, those transformations can be converted into the matrix forms and the resulting matrix is explicitly calculated as shown the following proposition.

Proposition 5. The degree reduction matrix for the four fixed Kharitonov’s DP curves associated with the original interval DP curve of degree $n$ into the corresponding four fixed Kharitonov’s DP curves of degree $n - 1$ can be given by:

\[ R_{n-1}^n = (F_n)^T(\frac{1}{n})^{T^T}(F_n)^{-1} \]

The resulting matrix of DP multiple degree reduction can be determined by equation (28) in proposition 6.

Proposition 6. The multiple degree reduction matrix for the four fixed Kharitonov’s DP curves associated with the original interval DP curve of degree $n$ into the corresponding four fixed Kharitonov’s DP curves of degree $n - k$ can be expressed by:

\[ R_{n-k}^n = (F_n)^T(M_{n-k})^{T^T}(F_n)^{-1} \]

VII. NUMERICAL EXAMPLE

Consider the interval DP curve defined by five interval control points:

\[ [p_{i0}, p_{i+}] = ([1.85,1.95],[1.40,1.50]) \]
\[ [p_{j0}, p_{j+}] = ([2.70,2.75],[1.60,2.10]) \]
\[ [p_{j0}, p_{j+}] = ([3.95,4.35],[3.40,3.85]) \]
\[ [p_{j0}, p_{j+}] = ([6.25,6.75],[1.80,1.90]) \]
\[ [p_{j0}, p_{j+}] = ([7.00,7.15],[2.00,2.50]) \]

It is required to reduce the degree of the given interval DP curve defined by them to 3.

The reduced interval vertices \([\beta_i^*, \beta_j^*]_{i=0}^{n=0}\) of the given interval DP curve are obtained after applying the algorithm explained in section VI as follows:

\[ [\beta_0^*, \beta_3^*] = ([1.85,1.95],[1.40,1.50]) \]

\[ [\beta_1^*, \beta_2^*] = ([4.70,5.25],[3.50,4.55]) \]
\[ [\beta_2^*, \beta_3^*] = ([3.15,3.50],[2.80,3.65]) \]
\[ [\beta_3^*, \beta_4^*] = ([7.05,7.60],[2.00,2.50]) \]

Simulation results in Figure (1) shows the envelopes of the original interval DP curve and the reduced interval DP curve, respectively.

VII. CONCLUSIONS

In this paper, an algorithmic approach to degree reduction of interval DP curve is presented. DP curve is a new parametric curve representation of which properties are NTP basis functions, corner cutting algorithm and linear computational complexity. These properties indicate to the efficiency of evaluating the points on DP curves. Among several important properties of a non-rational curve, the degree reduction of a curve from $n$ into $n - k$ degrees is very useful for many applications to decrease the complexity of the polynomials. The proposed method of interval DP curve degree reduction is based on conversion matrices between Bezier and DP curves and Bezier degree reduction matrix. The four fixed Kharitonov’s polynomials (four fixed DP curves) associated with the original interval DP curve are obtained. These four fixed DP curves are transformed into four fixed Bezier curves. The degree of the four fixed Bezier curves is reduced based on the matrix representations of the degree reduction process. The process of degree reduction $k$ times are applied to the four fixed Bezier curves of degree $n$ to obtain the four fixed Bezier curves of degree $n - k$ without changing their shapes. The four fixed reduced Bezier curves are converted into DP curves of the same degree. Finally the reduced interval DP control points are obtained from the four fixed reduced DP control points. Comparing two bitmapped images in image processing and visualization needs to be compared from their pixels by matching pixel-by-pixel. Consequently, it takes a lot of computational time while the comparison of two vector-based images is significantly faster. Sometimes these raster graphics images can be approximately converted into the vector-based images by various techniques. After conversion, the problem of comparing two raster graphics images can be reduced to the problem of comparing vector graphics images. Hence, the problem of comparing pixel-by-pixel can be reduced to the problem of polynomial comparisons. In computer aided geometric design (CAGD), the vector graphics images are the composition of curves and surfaces.
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