Singularities Treatment in Solving Volume Electric Field Integral Equation over Tetrahedron Meshing

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Abstract—The method of moment solution of the volume integral equation suffers from singular volume integrals. When applying the gradient-gradient operator on the green’s function, it allows the choice of piecewise constant function for the expansion and testing functions. But it results in strong singularities of order \( \frac{1}{r^3} \). In this paper, robust and accurate technique based on the RWG function is used. The RWG function is used both as a testing and as a basis functions. The proposed approach divides the integrals into two parts, one slowly varying nonsingular part which can be integrated numerically over the volume of the tetrahedrons and a singular part which will be converted into surface integrals of integrands R and VR that has analytical solutions. Also, a proposed fast and accurate numerical approach is presented to solve the nonsingular integral part. Validation of the proposed technique is presented. Very good results are obtained.

Index Term— Volume Integral Equation, Method of Moments, Singular Integrals.

I. INTRODUCTION

The electromagnetic scattering problems from arbitrary shaped dielectric objects are of main interest due to its wide applications. Some of these applications are antenna design such as dielectric resonator antenna, microstrip antenna, and target identification problems. The surface integral equations (SIE) based techniques [1-6] for solving dielectric scattering problems depends on the surface equivalent electric and magnetic current distributions on the scatterer surface. This allows solving homogenous material distribution only. In additions, the problem solution suffers from complicated singular terms. On the other hand, the Volume Integral Equation (VIE) based techniques [7-12] efficiently solve inhomogeneous dielectric distribution with less complicated singular terms. In case of inhomogeneous distribution, VIE is more convenient and has less time of execution. On the other side the SIE consumes less time than VIE in the solution of large homogenous scattering problems. But with the advent of high performance techniques such as parallel processing using MPI on cluster or grid of computers [7], the problem of time consumption can be reduced. Ylä-Oijala, and Taskinen [8] introduced a volumetric cylindrical pulse basis function and point-matching Moment-Method procedure that formulate an eigenvalue problem. They introduced a singularity treatment based on Bessel function identities. Sancer et al. [9], identify where derivatives of a discontinuous function arise in the derivation of the volume representation. A tetrahedron based model is used to represent arbitrarily shaped scattering geometry [10] where the Gradient-Gradient operator is applied that results in a singularity of the order \( \frac{1}{r^3} \) and allows a pulse basis function usage.

In this paper, the volume integral equation (VIE) in conjunction with the method of moment (MoM) is applied to solve scattering from arbitrary shaped dielectric objects. The two main problems in the moment method solvers are the geometric modeling of the structure and the integrations that encountered with the MoM matrix filling. This paper focuses in a new trend in solving the problem of integration singularities that appear in the moment method matrix. The tetrahedron model is used to model the geometry of the object under consideration and the RWG basis function for the volume current density is applied. The volume integrals is converted into surface integrals of analytically solved singular terms of the order R and VR. In this paper, a summarization of the moment method solution for the volume integral equation [12] is introduced in a way that facilitates the application of the analysis as a computer algorithm taking into consideration our novel way of integration.

The paper is organized as follows: section II explains the problem formulation for the MoM algorithm utilizing our proposed numerical integration technique taking into consideration the singularity treatment. In section III, the results are introduced. Finally, section IV presents the conclusions for this research.

II. PROBLEM FORMULATION

A. The Method of Moments solution of the volume integral equation

Consider the following electric field integral equation [12]

\[
\begin{align*}
\frac{\partial \mathbf{E}^t}{\partial \mathbf{r}} + j\omega \mathbf{A}^t(\mathbf{r}) + \nabla \Phi(\mathbf{r}) &= \mathbf{E}^i(\mathbf{r}) \\
\Phi(\mathbf{r}) &= \frac{1}{4\pi\varepsilon_0} \int_V -\frac{\mathbf{v}^t(\mathbf{r})}{|\mathbf{r} - \mathbf{r}'|} d\mathbf{r}'
\end{align*}
\]

Where \( \mathbf{A}^t(\mathbf{r}) = \frac{\mu_0}{4\pi} \int_V \mathbf{j}^t(\mathbf{r}) e^{-j\kappa_0|\mathbf{r} - \mathbf{r}'|} d\mathbf{r}' \)

Eq. (1) in conjunction with Eqs. (2) and (3) constitute the volume electric field integral equation where \( \mu_0 \) and \( \varepsilon_0 \) are the
free space permeability and permittivity, respectively. \( \vec{I}(\vec{r}) \) is the volume current distribution in the dielectric material. \( \vec{D}(\vec{r}), \vec{A}(\vec{r}) \) and \( \vec{\phi}(\vec{r}) \) are the displacement vector, the magnetic vector potential and the scalar vector potential, respectively.

After applying the MoM procedure [12], the volume integral equation is reduced to a system of linear equations in the form \([Z][I] = [V]\), where the elements of the \( Z \) matrix can be summarized as follows

\[
\gamma_{mn} = A_{mn} + \Phi_{mn} + D_{mn}
\]

(4)

Where

\[
A_{mn} = \alpha_{m+n}^++\alpha_{m-n}^-+\alpha_{m+n}^-+\alpha_{m-n}^+
\]

(5)

\[
\Phi_{mn} = \beta_{m+n}^++\beta_{m-n}^-+\beta_{m+n}^-+\beta_{m-n}^+
\]

(6)

\[
D_{mn} = \zeta_{m+n}^++\zeta_{m-n}^-+\zeta_{m+n}^-+\zeta_{m-n}^+
\]

(7)

\[
\alpha_{m+n}^+ = \frac{-\omega^2\mu_0\varepsilon_0}{36\pi} k_0^2 \int_{V_m} \frac{\vec{E}_n^2}{R} \cdot \vec{e}^{jk\vec{R}_n^2} dV
\]

(8)

\[
\alpha_{m+n}^- = \frac{-\omega^2\mu_0\varepsilon_0}{36\pi} k_0^2 \int_{V_m} \frac{\vec{E}_n^2}{R} \cdot \vec{e}^{jk\vec{R}_n^2} dV
\]

(9)

where \( R_m^c = |\vec{r}_m^c - \vec{r}| \) and \( \vec{r}_m^c \) is the position vector to the centroid of the face number \( m \).

\[
\zeta_{m+n}^+ = \begin{cases} +\zeta_{mn}^+ & \text{if } T_m^+ = T_n^- \\ -\zeta_{mn}^- & \text{if } T_m^- = T_n^+ \\ 0 & \text{if } T_m^\pm \neq T_n^\mp \end{cases}
\]

(12)

\[
\zeta_{mn} = \int_{T_p} \vec{r}_n \cdot \vec{E}_m dV
\]

(13)

where \( T_p \) is a tetrahedron containing both \( \vec{r}_n \) and \( \vec{r}_m \). The surface integrals in (8) and (9) can be solved using the approach described in [14]. The volume integrals in (8), (9), and (10) can be viewed as two integrals. Each of them has singular integrand that needs special treatment.

### B. Integral Evaluation procedure

In this paper, the singular integrals over tetrahedron volumes that appear in the volume integral equation solution are evaluated analytically. Next, it will be shown that, the singular integrals will be reduced to integrals of the form \( \frac{1}{r} \) or \( \frac{1}{R} \) over the volume of tetrahedrons. Where \( R \) is the unit vector from an observation point to a source point.

In this part, we present the evaluation of the singular integral around a volumetric sphere of radius \( \delta \) then taking the limit as \( \delta \) tends to zero. This is to exclude the regions of singularity for a separate treatment. First, consider a distribution of sources within a tetrahedron shown in Figure (1). Assume a sphere inside the tetrahedron under consideration or in case of a point on any face, it may be considered half a sphere or a part of sphere in case of a corner point. The first integral is

\[
\lim_{\delta \to 0} \int_0^1 R d\Omega = \lim_{\delta \to 0} \int_0^1 \sin \theta \cos \theta d\Omega = 0
\]

(14)

where \( \Omega_x = \sin \theta \cos \phi \) and \( \Omega_y = \sin \theta \sin \phi \).

The second integral is

\[
\lim_{\delta \to 0} \int_0^1 R d\Omega = \lim_{\delta \to 0} \int_0^1 \frac{R^2}{\delta} \sin \theta dR d\Omega = 0
\]

(15)

Where

\[
\Omega_x = \left( \frac{\sin^2 \theta - \theta}{2} \right) \sin (\phi_2 - \phi_1)
\]

(16)

\[
\Omega_y = \left( -\frac{\sin^2 \theta - \theta}{2} \right) \cos (\phi_2 - \phi_1)
\]

(17)
\[ \Omega_z = \sin(\theta_2 - \theta_1) \]  

(18)

Fig. (1) The three possible regions that may include singular points

From the above analysis, it is seen that the integrals at the singular points has a definite value of zero. So, one can exclude these points from integration when integrating in the whole tetrahedron.

After the extraction of the singular points from the volume of integration, the volume integrals will be converted to surface integrals over the surfaces enclosing the volume of integration. In the Moment Method solution of the volume integral equation, two integrals need to be treated accurately as will be illustrated in the following two subsections

B.1. The First Integral

\[ l_0 = \int_{V_p} \frac{e^{-jrR}}{R} \, d\hat{v} \]  

(19)

The integral (19) can be divided into two parts

\[ l_0 = l_1 + l_2 \]  

(20)

where

\[ l_1 = \int_{V_p} \frac{e^{-jrR-1}}{R} \, d\hat{v} \]  

(21)

\[ l_2 = \int_{V_p} \frac{1}{R} \, d\hat{v} \]  

(22)

The integral (21) is non-singular integral and have slowly varying nature that could be performed numerically. The integral (22) is a singular integral that needs a regularization scheme that transforms it into a nonsingular integral. Figure (2) shows the variation of both singular and non-singular terms. It is evident that the non-singular term has a slaw variation, so it needs minimum number of integration points in the numerical integration. On the other side, the integrand in the singular term approaches infinity when the source and observation points coincide with each other.

After proving that, although the integrand \( \frac{1}{R} \) is singular when the source and observation points coincide with each other, its integral over a tetrahedron has a limited value. Its solution can be solved in the following manner; if one can find a vector function \( \vec{A} \) that has its divergence equals to the volume integrand \( \frac{1}{R} \), then the volume integral is converted to a surface integral using the divergence theorem [13] pp.67.

Since the function depends only on the radius \( R \) then the divergence of \( \vec{A} \) is written as

\[ \frac{1}{R^2} \frac{\partial}{\partial R} (R^2 A_R) = \frac{1}{R} \]  

(23)

By simple manipulations, one can get \( A_R \) yielding \( A_R = 0.5 \). Thus, the integral (22) can be rewritten as,

\[ l_2 = \int_{V_p} \frac{1}{R} \, d\hat{v} = 0.5 \int_{V_p} \hat{\nabla} \cdot (\vec{R}) \, d\hat{v} \]  

(24)

Since \( \hat{\nabla} \cdot (\vec{R}) = -\hat{\nabla} \cdot (\vec{R}) \), then equation (24) is written as

\[ l_2 = -0.5 \int_{V_p} \hat{\nabla} \cdot (\vec{R}) \, d\hat{v} \]  

(25)

Using the divergence theorem, the integral (25) is converted to surface integral as follows

\[ l_2 = -0.5 \int_{S_p} \hat{n} \cdot d\hat{s} \]  

(26)

Where \( S_p \) is the closed surface surrounds the volume of integration \( V_p \). The integral (26) is now nonsingular integral which can be evaluated numerically. But with simple manipulations, it can be solved analytically in the following form

\[ l_2 = -0.5 \hat{n} \cdot \int_{S_p} \hat{\nabla} R \, d\hat{s} \]  

(27)

The integral (27) has an analytical solution [14]. This analytical solution, works to transform the integral from an integral over a
surface into an integral over a line. This results in more accurate solution if the observation points appear on the surface of integration.

This is illustrated with the following example using the analytical solution compared to the numerical one. Suppose that the surface of integration is the triangle shown in Figure 3 that has the vertices (1,0,0,0,0), (-1,0,0,0,0), and (0,0,2,3,0,0). Also, suppose that the observation point lies at the centroid of the triangle. It can be noticed that the integrand is a unit vector i.e., the integration is a summation of unit vectors from all points in the source triangle to the centroid of the triangle. Since there is no component in the z direction, then the integration will be zero in the z direction. Thus, the integration will be in the x y plan. Since all points lies symmetrically around the y-axis then the fields will be vanished in the x direction. Then the integration will be in the y direction only.

\[ I_{0} = \int_{m'} \int_{p} \left( \hat{r} - r \right) e^{-jKR} d\hat{\nu} \]  

(28)

Where \( \hat{f}_{m} \) is the testing function which is the same as RWG basis function.

By comparing the results of both the numerical and analytical solutions, the results is as follows; 0.3811 in the x direction for the numerical integration versus zero with the analytical one and 86.65e-2 in the y direction versus 1.0169e-2 with the analytical one. The z-component is zero for both. From the previous results, one can notice that with 4.4e6 integration points in the numerical integration the results deviates obviously in both the x and y direction, so the need for analytical solution becomes necessary.

B.2. The Second Integral

\[ I_{0} = \int_{m'} \int_{p} e^{-jKR} d\hat{\nu} \]  

(29)

The integral (28) is rewritten in the following form

\[ I_{0} = \int_{m'} \int_{p} \hat{r} e^{-jKR} d\hat{\nu} - \hat{f}_{m} \int_{p} \hat{R} e^{-jKR} d\hat{\nu} \]  

(30)

The integral (30) can be rewritten as

\[ I_{0} = \hat{t}_{m'} \left( r - r \right) I_{0} - \hat{f}_{m} \int_{m'} \hat{R} e^{-jKR} d\hat{\nu} \]  

(31)

The integral (31) consists of two parts; the first is the \( I_{0} \), which is regularized using (21) and (27); the second integral which has the same problem as with equation (26). So, there is a need for analytical solution for that part.

Let, \( I_{4} = \int_{m'} \int_{p} e^{-jKR} R d\hat{\nu} = I_{3} + I_{4} \)  

(32)

Where \( I_{3} = \int_{m'} \int_{p} e^{-jKR} R d\hat{\nu} \)  

(33)

Where \( I_{3} \) has a slowly varying integrand, so the integral can be performed numerically and it will give sufficient accurate results.

And \( I_{4} = \int_{m'} \int_{p} R d\hat{\nu} = -\int_{m'} \hat{f}_{m} \hat{\nabla}R d\hat{\nu} \)  

(34)

Using the identity [13] pp.52,

\[ \hat{f}_{m} \hat{\nabla}R = \hat{\nabla}. (R \hat{f}_{m}) - R \hat{\nabla}. \hat{f}_{m} \]  

(35)

Since \( \hat{f}_{m} \) is a constant vector with respect to the primed coordinates then \( \hat{\nabla}. \hat{f}_{m} = 0 \)

Thus \( I_{4} = -\int_{m'} \hat{\nabla}. (R \hat{f}_{m}) d\hat{\nu} \)  

(36)

Applying the divergence theorem on equation (36) results in,

\[ I_{4} = -\hat{n}. \int_{m'} R d\hat{s} \]  

(37)

The integral (37) is a nonsingular integral that gives good results with numerical integration or using the analytical formula [14].

B.3. Numerical Integration Algorithm

In this section, a new numerical integration technique by volume subdivision is proposed. The proposed volume numerical integration by subdivision is performed by dividing the volume of integration (Tetrahedron) into sub-tetrahedrons. The integrand is calculated at the centroid of each sub-tetrahedron multiplied by its volume. Finally the integration will be the summation of those calculated values. The subdivision can be performed by dividing each tetrahedron member into equal parts, N subdivision. The sub-division
strategy is performed in three steps. The first step is to develop an algorithm for dividing a tetrahedron into $2^3$ sub-tetrahedrons. The second step is to develop an algorithm for dividing a tetrahedron into $3^3$ sub-tetrahedrons. According to the required subdivision ratio, one can decide which algorithm to use for example, if N equals 2 use the $2^3$ algorithm once, if N equals 3 use the $3^3$ once, if N equals 4 use the $2^3$ algorithm twice, if N equals 5, use the $2^3$ then the $3^3$ algorithm once, and so on.

The $2^3$ subdivision algorithm

The algorithm starts by calculating the mid points of the six edges of the original tetrahedron. The tetrahedron has 4 vertices. Each one connected to three edges. Each vertex can constitute a sub-tetrahedron with the 3 mid points of the 3 edges that is collected at this vertex. This is shown in Figure (4-a). In order to constitute the inner tetrahedrons, one can take any two of the preformed four tetrahedrons. By noticing the six midpoints of the main tetrahedron, one can find that only one point that does not exist in one of the selected tetrahedrons and that there are only two inner faces of these two sub-tetrahedrons. These two faces constitute two sub-tetrahedrons with the free point. Similar remarks are noticed for the other two sub-tetrahedrons as shown in Figure (4-b). The resulting total sub-tetrahedrons are $2^3$.

![Fig. 4. The subdivision of the tetrahedron into $2^3$ sub-tetrahedrons](image)

The $3^3$ subdivision algorithm

The Tetrahedron division can be viewed as two parts; the lower part is divided into 8 sub-tetrahedrons using the $2^3$ subdivision algorithm. The upper part has two parallel triangular faces. The distance between them is calculated by dividing the members that is not included in the face by 3. The upper triangle can be subdivided into $3^2$ sub triangles, while the lower can be subdivided into $2^3$ sub triangles using the algorithm [19]. A connection mechanism between the upper and lower faces should be done to build a set of 19 sub-tetrahedrons. This connection is done using visualization program written in Matlab 7.0 to check that there is no intersections between the resulting sub-tetrahedrons. The total sub-tetrahedrons equal $3^3$.

It is worth mentioning that all the sub-tetrahedrons either using the $2^3$ or the $3^3$ algorithms are congruent to the mother tetrahedron and all of them have the same volume.

III. RESULTS AND DISCUSSIONS

The main point in the implementation of the moment method is the evaluation of the integrals required to calculate the Z-matrix. The first test part is the evaluation of the efficiency of the numerical integration by applying the different numerical algorithms with the analytical solution. Let the source tetrahedron has the following vertices points; (0,2.3,0), (1.0,4.3,0), (3,1.3,0), and (2.,3.3,4) and the observation point has the value of 2 in the x direction and 3.3 in the y direction. Figures (5) and (6) show the integral value of $\int \frac{1}{R} d\vec{v}$, and the $z$–component of $\int \left( \vec{r} - \vec{r} \right) \frac{1}{R} d\vec{v}$ versus z, respectively. These two integrations are performed using both the proposed analytical solution and the numerical solution using the subdivision algorithms.

The numerical solution degradation may be alleviated by increasing the number of points of integration but this will increase the probability of the coincidence of the source and observation points and this will result in an infinity value for the integrals. So, the proposed solution will prevent the occurrences of infinity results and will increase the accuracy of the results without the headache of searching the best point that may result in acceptable results.

![Fig. 5. The integral (9) versus z.](image)
In order to test the accuracy of the subdivision algorithm, we use the previous problem of integrating $\int \frac{1}{R} d\phi$ but at this time at $z=2$. From Figure (7) depicts the relative error of the numerical integration relative to the analytical solution. It is noticed that the relative error is in the order of $10^{-3}$. So, it is recommended to use only the $3^3$ algorithm but in case of the need for more accurate results one can increase the subdivision ratio.

Electric field distribution inside an inhomogeneous dielectric cylinder due to an incident plane wave.

The first example is an inhomogeneous finite cylinder of radius 2 cm and the height of 4 cm. The cylinder has four horizontal layers, each with different dielectric constant as shown in Figure (8). The height of each layer is 1 cm. Relative dielectric constant varies from 10 for the top section to 2.5 for the bottom section, in steps of 2.5. The frequency of the $z$-polarized plane wave incident in the positive $x$-direction is 750 MHz. The scattered field in the dielectric has the dominant $z$-component, is compared to the Ansoft HFSS solution [15] where a very good accuracy is noticed. The field is evaluated on the cylinder axis $x = 0$ cm, $y = 0$ cm. Figure (9) shows the scattered electric field as a function of the distance along the axis of the cylinder depicted in Figure (8), with 576 tetrahedrons and 1264 faces.

Electric field distribution inside a dielectric sphere due to an incident plane wave.

The second example is a homogeneous dielectric sphere of radius 50 cm, $\varepsilon_r=36$. The sphere has a 512 tetrahedron with 1088 faces as shown in Figure (10). The dominant scattered electric field inside the sphere is calculated along the $z$ axis. Results are compared to the Mie series solution published in [12] see Figure (11). One can notice the good consistence with the published data even with the low resolution used in the test.


IV. CONCLUSION

In this paper, detailed analysis of the singular triple integrals in the solution of the volume integral equation is introduced. The singular integrals are investigated where good accuracy is observed. A new numerical integral is also investigated and compared at different resolutions where good results are observed. However, the far field results are not sensitive to minor errors in the internal field distribution. Therefore, a more critical check on the accuracy of the calculations is the accuracy of the internal fields. The VIE solution is tested by calculating the internal fields inside the dielectric structures where good accuracy is noticed.

REFERENCES


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