Mixed Sensitivity $H_2/H_\infty$ Control of a Flexible-Link Robotic Arm

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Abstract— Dynamics of multi-link manipulators with flexible links include complex high-order equations which make their control problem very challenging. This paper presents a new method for simultaneous motion and vibration control of a two-link flexible manipulator using $H_2/H_\infty$ control. A multi-element finite element model of the manipulator is casted into the generalized plant model, and $H_2$ and $H_\infty$ controllers are synthesised employing the LMI-based optimization algorithm of MATLAB. Finally, a mixed sensitivity $H_2/H_\infty$ control is proposed based on an $H_2$ norm constrained $H_\infty$ control design. It is shown that the method of control design can be used successfully for the control of the joint parameters as well as suppression of vibration at the tip (end-effector) of the manipulator.

Index Term-- Flexible manipulators; motion and vibration control; mixed $H_2/H_\infty$ control; finite element modelling

I. INTRODUCTION

Flexible-link manipulators (FLMs) originally were developed to achieve advantages such as lightness, high payload capacity, accessibility to wider workspaces, and so on. These advantages are of particular importance in space robots [1]. In order to achieve the advantages, robotic arms can be designed with long and slender links; which in turn introduce flexibility to the system due to the elastic behavior such as bending and torsion of the links. Abundant research has been done to cope with modeling and control of the FLMs; and it is yet an open field.

Many methods have been proposed for modelling the dynamics of FLMs. Some famous examples include the assumed mode method (AMM), finite element method (FEM), perturbation methods, and Ritz expansion. Some researchers have reported studies on the dynamics of FLMs, without proposing a controller for the manipulator; while others propose methods for control, because most researchers are interested in models that can be used in control design [2]. Dwivedy and Eberhard [3] represented modeling of a two-link FLM using AMM with four modes. Supriyono and Tokhi [4] proposed a model of a single-link FLM based on biologically-inspired optimization technique of bacterial foraging algorithms.

The dynamics of a FLM has challenging complexities for control engineers. Many research were carried out attacking the complexities in the control of the FLMs. Chapnik, et al. [5] proposed an open-loop control system using frequency-domain techniques to compute a desired hub torque profile. Khorrami, et al. [6] studied a feedback linearization method for control of a two-link FLM. Bai, et al. [7] studied identification of friction in a two-link FLM. An inversion based controller that cancels the effect of the unstable zeros in a single-link FLM was presented in [8]. Cole and Wongratanaphisam [9], suggested an adaptive method for feed-forward control of a two-link FLM to achieve zero residual vibration in rest-to-rest motion. Shawky et al. [10], studied nonlinear control using end-point position feedback for a single-link FLM. Feliu et al. [11], studied passivity-based control of a single-link FLM considering robustness against payload variations and friction of the joints.

Perhaps the most important control problem in the FLMs is vibration suppression during or after a maneuver of the manipulator. In order to control the vibrations, we will need to have a standard norm that shows the severity of vibration. Depending on how such ‘norm’ is defined, the objectives and methods of control synthesis will vary. Two common tools to quantify vibration levels are the $H_2$ and $H_\infty$ norms. Then, a controller that minimizes the $H_2(H_\infty)$ norm of an output signal is known as $H_2$ ($H_\infty$) controller. It is known that noise or random disturbances are more naturally expressed in $H_2$ or RMS terms. An LQR/LQG controller in fact attacks the $H_\infty$ performance aspects, and therefore is a good candidate for controlling the seemingly random vibrations of flexible systems. Konno and Uchiyama [12] studied LQR control of a two-link manipulator. Milford and Asokanthan [13] also used an LQG controller.

Due to uncertainties of the dynamic model, robust control of the FLMs has received particular attention in the literature. Along with various robust control methodologies, $H_\infty$ control provides methods to deal with the stability and performance robustness. As some examples, Hisseine and Lohmann [14] considered sliding mode as well as nonlinear $H_\infty$ control of a single-link FLM. Ming-Tzu and Yi-Wei [15], employed the $H_\infty$ framework for achieving good performance of PID control for a single-link FLM.

In practical applications, sometimes standard $H_\infty$ synthesis methods are not adequate to capture all design specifications. Therefore imposing an additional $H_2$ performance requirement to the $H_\infty$ synthesis may include the advantages of an LQG design. Likewise, including an $H_\infty$ performance requirement can improve LQG design [16]. Banavar [17] added an $H_\infty$ controller to an LQG to improve stability of the LQG control of a single-link FLM. [18] designed a mixed-sensitivity $H_\infty$ controller for a single-link FLM including gravity terms.

The Mixed-Sensitivity $H_2/H_\infty$ Control has been used successfully for various control engineering applications. Safonov, et al. [19] used a mixed-sensitivity $H_\infty$ control for robust control of a large scale space structure. Ohishi, et al. [20] proposed a force control technique for a manipulator

In this work, the problem of $H_2/H_\infty$ control design for a planar two-link manipulator with flexible links is investigated. For this aim, first two controllers, i.e. one $H_2$ and one $H_\infty$ controller, are designed and compared in terms of speed (settling time) and level of vibration of the tip of the manipulator. Then, the final mixed $H_2/H_\infty$ controller is achieved by solving the $H_\infty$ optimization problem under $H_2$ constrains. The model used in this work is a multi-element FEM model of the manipulator. The dynamic equations of FEM models with multiple elements have big matrices which make the feedback control design more complex. The numerical method for $H_2/H_\infty$ control design using linear matrix inequality LMI which is provided in MATLAB, was employed successfully for the control design.

The remainder of this paper is organized as follow. First, in section 2, the model of the two-link FLM is introduced. In section 3, the methodology of the $H_2/H_\infty$ LMI optimization is represented. Section 4 is devoted to $H_2$ control design which will be used as the first cut or reference to evaluate the performance of control system in terms of suppression of vibration. Then in section 5, an $H_\infty$ controller is presented. In section 6 the mixed $H_2/H_\infty$ controller is proposed to make a trade-off between the advantages of the $H_2$ and the $H_\infty$ control. Finally, the conclusions will be summarized.

II. THE MODEL OF THE FLEXIBLE MANIPULATOR

In this paper, similar to [28], FEM is used for modeling the two-link FLM. For this reason each link is divided into ten Euler-Bernoulli beam elements. The FEM model of the manipulator is shown in Figure 1. Denoting the degrees of freedom of each node $i$ by $V_i$, $\Phi_i$, which show the linear and angular displacements of the node; and showing the joint angles with $\theta_1, \theta_2$, the vector of generalized coordinates can be written as

$$ q = [q_1, q_2, q_3, ..., q_{10}]^T $$

The superscript $^T$ stands for transpose of the vector. The partitioning line in equation (1) separates the coordinates of the first and the second links.

The dynamic equations of the system can be written as

$$ \frac{\partial \dot{q}}{\partial \dot{x}} = \frac{\partial \Phi}{\partial \dot{x}} + \frac{\partial \Psi}{\partial \dot{x}} $$

$$ \frac{\partial \Phi}{\partial x} + \frac{\partial \Psi}{\partial x} $$

where $\mu$ is the mass per unit length, and $\dot{\theta}_1$ and $\dot{\theta}_2$ are the time derivatives of the vector of position of a particle on the first and the second link, as shown in Figure 1. Next, supposing $EI_1$ and $EI_2$ are the flexural rigidity of the first and the second link, the potential energy of the system can be obtained as

$$ U = \sum_{i=1}^{10} \frac{1}{2} EI_1 \left( \frac{\partial^2 v}{\partial x^2} \right) dx + \sum_{i=1}^{10} \frac{1}{2} EI_2 \left( \frac{\partial^2 \psi}{\partial x^2} \right) dx $$

In order to derive the dynamic equations, the potential and kinetic energy of the system is measured by summation of energies of all the Euler-Bernoulli elements. For this aim, the elemental values of the kinetic and strain energies of each element are measured using integration over Hermite shape functions. The kinetic energy will be then

$$ T = \sum_{i=1}^{10} \frac{1}{2} EI_1 \left( \frac{\partial^2 v}{\partial x^2} \right) dx + \sum_{i=1}^{10} \frac{1}{2} EI_2 \left( \frac{\partial^2 \psi}{\partial x^2} \right) dx $$

$$ \frac{1}{2} EI_1 \left( \frac{\partial^2 v}{\partial x^2} \right) dx + \frac{1}{2} EI_2 \left( \frac{\partial^2 \psi}{\partial x^2} \right) dx $$

In order to fulfill the integration, according to the Euler-Bernoulli beam theory, a matrix of Hermite shape functions, $N(x)$, is adopted which relates the continuous function $v(x)$ of an arbitrary element $i$ to the nodal coordinates as follow

$$ v(x,i) = [N(x)] [v_i, \varphi_i, \psi_i, \theta_{i,1}, \theta_{i,2}]^T $$

Now, using equation (4) in equations (2) and (3), and considering the physical parameters of the system as given in
Table I, the overall energies are obtained in terms of the generalized coordinate vector of equation (1).

\begin{table}[h]
\centering
\begin{tabular}{|l|c|c|c|}
\hline
\textbf{Parameter} & \textbf{Upper arm} & \textbf{Forearm} & \textbf{Unit} \\
\hline
Length & 300 & 300 & mm \\
Thickness & 1.5 & 1 & mm \\
Width & 30 & 25 & mm \\
Mass per unit length & 1.993 & 1.107 & g/cm \\
Elasticity Modulus & 113.8 & 113.8 & GPa \\
Initial angle & \pi / 6 & \pi / 3 & Rad \\
\hline
\end{tabular}
\caption{Physical parameters of the manipulator}
\end{table}

Defining the inertia and stiffness matrices \( M \) and \( K \), the overall kinetic and potential energies can be rearranged in a matrix form as

\[ T = \frac{1}{2} \dot{q}^T M \dot{q} \quad ; \quad U = \frac{1}{2} \dot{q}^T K \dot{q} \quad (5) \]

Having measured the system energies, the Lagrange’s equations may be used to yield the relation between the first and the second motors (\( \tau_1, \tau_2 \)) and \( q \) as

\[ \sum_{k=1}^{n} \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_k} \right) - \frac{\partial L}{\partial q_k} = Q_k \quad (6) \]

In which \( L = T - U \) is the Lagrangian of the system. With a ‘\( n \) by 2’ matrix \( F \) known as the input matrix given by

\[ \sum_{k=1}^{n} Q_k \delta q_k = \dot{\omega}_w = \sum_{k=1}^{n} (F_{s_k} \tau_1 + F_{s_2} \tau_2) \delta q_k \quad (7) \]

The generalized forces \( Q_k \) are related to the virtual work done by non-conservative forces and the torques applied by the first and the second motors (\( \tau_1, \tau_2 \)). Then, the linearized form of the dynamic equations is represented in the following matrix equation

\[ M \ddot{q} + K \dot{q} = F [\tau_1, \tau_2]^T \quad (8) \]

Equation (8) needs to be supplemented with the relevant boundary conditions (BCs). For the manipulator, the BCs include pinned BC at the joints, and free BC at the tip. Therefore, for the first node of each link just rotational DOF are possible; and so some of the columns and rows of \( M, K \), and \( F \) can be eliminated, and the matrices are reduced to \( \tilde{M}, \tilde{K}, \) and \( \tilde{F} \) and equation (8) is rewritten as follow

\[ \tilde{M} \ddot{q} + \tilde{K} \dot{q} = \tilde{F} [\tau_1, \tau_2]^T \quad (9) \]

Then, with zero and identity matrices \( O \) and \( I \), a state space representation of the system is obtained as

\[ \dot{X} = AX + Bu, \]

\[ X = \begin{bmatrix} \dot{q} \\ \dot{\dot{q}} \end{bmatrix}, \quad A = -M^{-1}K \quad I \quad B = \begin{bmatrix} O \\ M^{-1}F \end{bmatrix}, \quad u = \begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix} \quad (10) \]

The transfer matrix of the system is also achieved by selecting the proper output vector of the joint angles and displacement of the tip. Then, the system is represented as

\[ G = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \\ G_{31} & G_{32} \end{bmatrix} \]

so that

\[ \begin{bmatrix} \theta_1 \\ \theta_2 \\ v_{\text{tip}} \end{bmatrix} = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \\ G_{31} & G_{32} \end{bmatrix} \begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix} \quad (11) \]

where \( \theta_1 \) and \( \theta_2 \) are angular motion of the shoulder and elbow joints, and \( v_{\text{tip}} \) is the deflection of the tip.

\[ \text{III. THE METHOD OF } H_2/H_\infty \text{ CONTROLLER DEIGN USING LMI SOLVER} \]

The method of mixed \( H_2/H_\infty \) synthesis using Linear Matrix Inequality (LMI) [29] (also developed in [30], and [31]) performs multi-objective output-feedback synthesis to design a suboptimal LTI controller \( K(s) \) that minimizes a mixed \( H_2/H_\infty \) criterion using convex optimization. First, the system equation is rearranged in the form of a generalized plant, \( P(s) \), with state vector \( x \), exogenous input \( w \), and control input \( u \) as follow

\[ P(s) : \begin{bmatrix} \dot{x} \\ z_e \\ z \end{bmatrix} = \begin{bmatrix} A_x & B_{w_x} & B_{u_x} \\ C_{x,z} & D_{w,z} & D_{u,z} \\ C_x & D_{w,y} & D_{u,y} \end{bmatrix} \begin{bmatrix} x \\ z_e \\ z \end{bmatrix} \]

Here \( y \) is the feedback signal, and \( z_e \) and \( z \) are the output signals used as performance index. The matrices \( A, B, C, \) and \( D \) are the system matrices. Then, a linear controller is considered as

\[ K(s) : \begin{bmatrix} \hat{z} \\ u \end{bmatrix} = \begin{bmatrix} A_k z + B_k y \\ C_k \hat{z} + D_k y \end{bmatrix} \quad (13) \]

Note that \( K(s) \) has the measured outputs of the plant as its input, and the input vector of the plant as its output. The corresponding closed-loop system can be measured and rearranged in the form of an LTI system as
\[
\begin{align*}
\dot{x}_m &= A_{\text{m}}x_m + B_{\text{m}}w \\
\dot{z}_m &= C_{\text{m}}x_m + D_{\text{m}}w \\
\dot{z}_2 &= C_{\text{cl}}x_2 + D_{\text{cl}}w
\end{align*}
\] (14)

The general form of the closed-loop system is sketched in Figure (2).

Note that with this configuration, the system has two output and one exogenous input vector. Therefore, two transfer functions are required to relate the outputs to the input. The closed-loop transfer functions from \( w \) to \( z_{\text{m}} \) and \( z_2 \) are denoted by \( T_{\text{m}}(s) \) and \( T_2(s) \) respectively.

That is \( [z_{\text{m}}, z_2] = [T_{\text{m}}, T_2] \{w\} \). Then, denoting by \( \| \cdot \|_\infty \) and \( \| \cdot \|_F \) the \( H_\infty \) norm, and \( H_2 \) norm of the transfer functions, the control objective of the \( H_2/H_\infty \) is to minimize \( \gamma, \nu \) so that

\[
\|T_{\text{m}}\|_\infty < \gamma, \quad (15 \text{a})
\]

and

\[
\|T_2\|_F < \nu, \quad (15 \text{b})
\]

The mixed \( H_2/H_\infty \) method \([31]\), seeks for a common Lyapunov matrix \( \chi = \chi_{\text{m}} = \chi_{\text{cl}} > 0 \) which satisfies a set of the multiple constraints using convex optimization of an LMI. The method relies on two theorems; the first stating that the closed-loop RMS gain from \( w \) to \( z_{\text{m}} \) does not exceed \( \gamma \), if and only if there exist a symmetric positive definite matrix \( \chi_{\text{m}} > 0 \) such that

\[
\begin{pmatrix}
A_{\text{m}} + \chi_{\text{m}} A_{\text{m}}^T & B_{\text{m}} & C_{\text{m}}^T \\
B_{\text{m}}^T & -I & D_{\text{m}}^T \\
C_{\text{cl}} x_{\text{m}} & D_{\text{cl}} & -\gamma I
\end{pmatrix} < 0
\] (16)

The second theorem, for satisfying the \( H_2 \) performance requirement, states the transfer function from \( w \) to \( z_2 \) does not exceed \( \nu \) if and only if \( D_{\text{cl}} = 0 \) and there exist symmetric matrices \( \chi_{\text{cl}} \) and \( Q \) such that

\[
\begin{pmatrix}
A_{\text{cl}} x_{\text{cl}} + \chi_{\text{cl}} A_{\text{cl}}^T & B_{\text{cl}} \\
B_{\text{cl}}^T & -I
\end{pmatrix} < 0
\] (17)

In this work, first one \( H_2 \) control and one \( H_\infty \) control are designed using the algorithm. Then, the final mixed \( H_2/H_\infty \) controller is achieved by solving a constrained \( H_\infty \) optimization problem to make a trade-off between the \( H_2 \) and the \( H_\infty \) control.

The architecture of the control system and the outputs considered for optimization is shown in Figure 3. In order to conform to the standard format given in Figure 2, the configuration of the controlled system is rearranged to the linear fractional transformation (LFT) as shown in Figure 4. These figures will be referred to in the next sections, to show the performance criterion.

IV. \( H_2 \) OPTIMAL DESIGN

One measure for vibration level is provided by the \( H_2 \) norm. The \( H_2 \) norm of a stable continuous system is related to the root-mean-square (RMS) of its impulse response. The \( H_2 \) norm also represents the steady-state covariance (or power) of the output response to unit white noise inputs. Therefore the objective of vibration suppression can be translated to
minimizing the $H_2$ norm of the output signals of a mechanical system. The controller design method with the objective of minimizing a $H_2$ norm is known as the $H_2$ optimal design. In this section an $H_2$ control design is performed for the two-link FLM. The design will be used as a reference for evaluation of the next $H_2/H_\infty$ controllers.

Let $T_2$ be the transfer function from the vector of desired joint angles to the output vector, including the actual joint angles as well as the transversal displacement of the tip due to elastic deformation.

$$
\begin{bmatrix}
\theta_1 \\
\theta_2 \\
v_{\text{tip}}
\end{bmatrix}
= T_2
\begin{bmatrix}
\theta_1^{\text{desired}} \\
\theta_2^{\text{desired}}
\end{bmatrix}
$$

(20)

The $H_2$ controller is designed targeting at the following optimization problem

$$\text{Minimize: } \| v_0 \|$$

(21)

The step response of the closed-loop system is shown in Figure 5. As the system has two inputs, two unit step commands are applied separately; and the response of three outputs are plotted. Note that the time range of the third output, that is the normal acceleration of the tip due to deflection, is different. The simulation result shows that the response of the joint angles is very slow; and the overshoot is more than 20%. However, the vibration of the tip of the manipulator has a very small value, in better words an optimal value in the sense of $H_2$ norm. The achieved optimal value for Equation 21 was: $v_0 = 0.3930$. This value will be used in next sections, as a reference point in our $H_2/H_\infty$ controller design.

V. THE MIXED SENSITIVITY $H_\infty$ OPTIMAL DESIGN

In addition to the $H_2$ norm, a well-known standard measure for evaluation of vibration levels is provided by the $H_\infty$ norm which can be thought of as a measure for the peak amplitude of the FRF of the system. This norm is even more common in vibration measurement and control as it shows the resonant characteristics of vibration modes. The controller design methodology targeting at the minimization of the $H_\infty$ norm is known as $H_\infty$ optimization design.

As mentioned before, different performance requirements of the controlled system can be translated to conditions on the $H_\infty$ norm of the $S$, $T$, and $KS$, where $S$ and $T$ are the sensitivity and complementary sensitivity transfer matrices, and $K$ is the controller. For example good tracking, noise attenuation and robust stability (with respect to multiplicative output uncertainties) can be achieved by small $T(s)$. On the other hand, the performance of the system in terms of command tracking and disturbance attenuation can be translated to requirements for $S(s)$ since it relates the error signals with references and disturbances. Additionally, the optimization of the control effort within a limited bandwidth (constraining actuator saturation) is equivalent to minimizing $KS(s)$. Thus, all in all, the optimization problem is summarized as a mixed-sensitivity synthesis aiming at minimization of the $H_\infty$ norm of

$$T_\infty = \begin{bmatrix} w_1 S \\
w_2 KS \\
w_3 T \end{bmatrix}$$

(22)

With weighting filters $w_{1,2,3}$, this objective is known as the weighted $S/KS/T$ (read $S$ over $KS$ over $T$) mixed-sensitivity. In this section the pure $H_\infty$ optimal design of the controller is considered. The controller is designed to

$$\text{Minimize: } \left[ \begin{array}{c} S \\ K S \\ T \end{array} \right]$$

(23)

The step response of the resulting closed-loop system with the $H_\infty$ controller is represented in Figure 6. It is observed that the $H_\infty$ provides better performance in terms of the rise time and overshoot of the step response, compared with the $H_2$ controller.

Fig. 5. Step response of the $H_2$ optimal design
Comparison of Figure 5 with Figure 6 reveals that the $H_2$ control results in less vibration at the tip. Therefore, it is inferred that including an $H_2$ condition in the $H_\infty$ norm optimization may provide better vibration suppression. The method of designing controllers with minimizing both the $H_2$ and $H_\infty$ norms are known as mixed $H_2/H_\infty$ design.

VI. THE MIXED $H_2/H_\infty$ DESIGN

In this section, a mixed $H_2/H_\infty$ design is proposed through the following optimization problem:

$$
Minimize: \quad \gamma = \frac{1}{\|K_S\|_\infty},
$$

Subject to: $\|P\|_2 \leq n v_0$.

The design parameter $n$ is used as a tool for tuning the controller. By increasing $n$ the controller performs similar to the $H_\infty$ controller; and decreasing $n$ will result a performance like the $H_2$ controller. Recall that $H_2$ controller was slower but had less vibration. Figure 7 shows the response of the controlled system with different values of $n$. The results show that with increasing the $n$, the response of the joint angles become more desirable. Particularly in an approximate range of $2 < n < 10$ the mixed control method is effective. With higher values of $n$, more overshoot and more vibration of the end-effector is resulted. Therefore, the final design is selected with $n=5$. Figure 8 show the step response of the closed-loop system with the final controller.

VII. CONCLUSION

In this paper a set of $H_2/H_\infty$ controllers was developed for simultaneous motion and vibration control of a two-link manipulator with flexible links. A multi-element FEM model of the manipulator was considered and the control problem was casted into standard configuration of linear fractional transformation. The simulation results showed that although the response of the closed-loop system under the $H_2$ control was very slow compared with that of the $H_\infty$ controller; the $H_2$ controller was more successful in vibration suppression. It was inferred that attacking at the $H_2$ norm is very effective in vibration control of the manipulator. Therefore, a mixed $H_2/H_\infty$ control was designed to include the advantage of the $H_2$ control in the $H_\infty$ design. It was shown that the mixed $H_2/H_\infty$ method provides a trade-off among the advantages of the $H_2$ and $H_\infty$ control for the complex dynamic system.

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