On group rings with involution

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Abstract. In this note, we consider the involution group ring $AG$ of an involution group $G$ over an involution ring $A$ with identity and prove the involution version of a Theorem due to Connell, which characterizes *-artinian group rings with identity. Furthermore, *-simple involution group rings are also investigated.

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1 Introduction

We consider only associative rings. An involution ring $A$ is a ring with involution * subject to the identities

$$a^{**} = a, (ab)^* = b^*a^*, (a + b)^* = a^* + b^*,$$

for all $a, b \in A$. Thus, the involution is an anti-isomorphism of order 2 on $A$.

By an involution group we mean a multiplicative group with involution satisfying the first two identities. Every group $G$ has at least one involution, namely the unary operation of taking inverse; that is $g^* = g^{-1}$, for every $g \in G$.

Recall that a biideal $B$ of a ring $A$ is a subring of $A$ satisfying $BAB \subseteq B$. An ideal (biideal) $I$ of an involution ring $A$ is called *-ideal (*-biideal), if it is closed under involution; that is $I^* = I$.

By the way, in the theory of involution rings, *-biideals have been used successfully (instead of one-sided ideals) in describing their structure (see for instance [1] and [3]).

Recall that a ring $A$ is said to be simple if $A^2 \neq 0$ and $A$ has no proper nonzero ideals. Analogously, an involution ring $A$ is called *-simple if $A^2 \neq 0$ and $A$ has no proper nonzero *-ideals. Obviously, a simple involution ring is *-simple while the converse is not true. For example, if $A$ is a simple ring and $A^{op}$ is its opposite ring then the ring $R = A \oplus A^{op}$, under the exchange involution * defined by $(a, b)^* = (b, a)$ for every $(a, b) \in R$, is *-simple but not simple (see [2]).

Finally, an involution ring $A$ is said to be *-artinian (*-noetherian) if it satisfies dcc (acc) on *-biideals (see [1] and [3]).

The following results due to Beidar and Wiegandt [3] and Aburawash [1] are useful in proving our main result.
Proposition 1 ([3], Theorem 3). If a ring $A$ with involution satisfies dcc on *-biideals, then its Jacobson radical $\mathfrak{Z}(A)$ satisfies dcc on subgroups and hence $\mathfrak{Z}(A)$ is nilpotent.

Proposition 2 ([3], Corollary 4). A ring $A$ with involution has dcc on *-biideals if and only if $A$ is an artinian ring with artinian Jacobson radical.

Proposition 3 ([1], Corollary 1). If $A_1, A_2, \ldots, A_n$ are *-artinian (*-noetherian) involution rings, then so is their direct sum $A_1 \oplus A_2 \oplus \ldots \oplus A_n$.

2 Group rings with involution

Let $A$ be a ring with identity and let $G$ be a group. Following [5], the group ring $AG$ consists of all finite linear combinations $\sum_{g \in G} a_g g$, $a_g \in A$, $g \in G$, where only finitely many coefficients $a_g$ are nonzero. In $AG$, addition is defined componentwise while multiplication is an extension of that in $G$. Furthermore, $AG$ has identity and $A$ is a subring of $AG$.

For Baer radical, Lemma 73.4 in [5] shows that the prime radical $(A)$ of the ring $A$ is a subring of the prime radical $\beta(AG)$ of the group ring $AG$.

Proposition 4 ([5], Lemma 73.4). If $AG$ is the group ring of a group $G$ over a ring $A$, then $\beta(A) = A \cap \beta(AG)$.

Moreover, the following result due to T. Connell gives a necessary and sufficient condition for a group ring with identity to be artinian.

Proposition 5 ([5], Lemma 73.11). Let $AG$ be the group ring of a group $G$ over a ring $A$ with identity. Then $AG$ is artinian if and only if $A$ is artinian and $G$ is finite.

Now, let $A$ be an involution ring with involution $\triangledown$ and $G$ be a group with involution $\hat{\triangledown}$. One can make $AG$ an involution ring by defining an involution $*$ on the group ring $AG$ in a natural way by $(\sum_{g \in G} a_g g)^* = \sum_{g \in G} a_g^\triangledown g^\hat{\triangledown}$, $a_g \in A$, $g \in G$. However, we can simply denote all the given involutions by $*$, since it will not lead to ambiguity.

Our main goal is to check whether, for involution rings with identity, descending chain condition on *-biideals is transferred between the group ring and its underlying ring. This is, however, the involutive version of T. Connell’s Theorem given in Proposition 4 (see also [4]).

Theorem 6 Let $AG$ be an involution group ring of a group $G$ over a ring $A$ with identity. Then $AG$ is *-artinian if and only if the ring $A$ is *-artinian and the group $G$ is finite.
Proof. If $AG$ is *-artinian then it is artinian and $\exists(A)$ satisfies dcc on subgroups, by Proposition 1. Hence, from Proposition 5, it follows that $A$ is artinian and $G$ is finite. $A$ and $AG$ are both artinian, so $\exists(A) = \beta(A)$ and $\exists(AG) = \beta(AG)$, whence by Proposition 4, $\exists(A) \subseteq \exists(AG)$. Thus $\exists(A)$ satisfies dcc on subgroups since $\exists(AG)$ does, and consequently $\exists(A)$ is artinian. Applying Proposition 2, $A$ would be *-artinian. For sufficiency, let $A$ be *-artinian and $G$ be finite. Since $AG$, as a left $A$-module, is a direct sum of $|G|$ copies of $A$, then $AG$ is *-artinian, according to Proposition 3.

For acc on *-bideals, we have

**Proposition 7** If the involution ring $A$ is *-noetherian and the group $G$ is finite, then the involution group ring $AG$ is *-noetherian.

**Proof.** Let $A$ be *-noetherian and $G$ be finite. Since $AG$, as a left $A$-module, is a direct sum of $|G|$ copies of $A$, then $AG$ is *-noetherian, by Proposition 3.

3 *-Simple Group rings with involution

The characterization of *-simple involution rings was given in [2] as follows.

**Proposition 8** ([2], Lemma 1). An involution ring $A$ is *-simple if and only if either $A$ is simple or $A = I \oplus I^*$, with $I \triangleleft A$ and $I$ is a simple ring, and the involution is the exchange involution $\vee$ defined by $(a, b^*)\vee = (b, a^*)$, for all $a, b \in I$.

The following result gives a sufficient and necessary condition for a group ring to be simple.

**Proposition 9** A group ring $AG$ is simple if and only if the ring $A$ is simple and the group $G = \{1\}$.

**Proof.** For necessity, let $AG$ be simple. From [5], page 313, we have

$$\omega G = \{x \in AG \mid x = \sum_{g} a_g g \text{ with } \sum_{g} a_g = 0 \} \triangleleft AG \text{ and } A \cong AG/\omega G.$$  

If $G \neq \{1\}$, then $0 \neq y = a1 - ag \in \omega G$ implies $\omega G = AG$, whence $A = 0$, contradicts $AG \neq 0$. Thus $G = \{1\}$ and consequently $\omega G = 0$. Hence $A \cong AG$ is simple. The sufficiency is obvious.

Finally, using Propositions 8 and 9, we get the following characterization of *-simple involution group rings.

**Theorem 10** An involution group ring $AG$ is *-simple if and only if the ring $A$ is *-simple and the group $G = \{1\}$.

**Proof.** Let $AG$ be *-simple. Since $\omega G$ is a *-ideal of $AG$, then either $\omega G = AG$ or $\omega G = 0$. The first case is impossible because $A \cong AG/\omega G$ implies $A = 0$, a contradiction. The second case implies, from $A \cong AG/\omega G$, that $A \cong AG$ is
*-simple and consequently $G = \{1\}$. For the converse, let $A$ be *-simple and $G = \{1\}$, then according to Proposition 8, either $A$ is simple or $A = I \oplus I^*$, with $I \triangleleft A$ and $I$ is a simple ring. If $A$ is simple then, by Proposition 9, $AG$ is simple and so *-simple. If $A$ is not simple, then the group ring $AG \cong A = I \oplus I^*$ is *-simple. ■

References


