Degree Elevation of Interval Bezier Curves

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Abstract—This paper presents a simple matrix form for degree elevation of interval Bezier curve. The four fixed Kharitonov’s polynomials (four fixed Bezier curves) associated with the original interval Bezier curve are obtained. The method is based on the matrix representations of the degree elevation process. The process of degree elevations \( k \) times are applied to the four fixed Bezier curves of degree \( n \) to obtain the four fixed Bezier curves of degree \( n + k \) without changing their shapes. Finally the new interval vertices \( \left[ \left[ B_i, B_i^* \right] \right]_{i=0}^{n+k} \) of the new interval polygon are obtained from vertices of the new fixed polygons of the four fixed Bezier curves. An illustrative example is included in order to demonstrate the effectiveness of the proposed method.

Index Term—Image processing, computer graphics, CAGD, degree elevation, interval Bezier curves.

I. INTRODUCTION

Computer Aided Geometric Design (CAGD) is a sort of computer-based application used for modeling and constructing various kinds of geometric objects, such as curves, surfaces, and solids. Those geometric entities can be explicitly formed in terms of the mathematical equations. Nowadays, the geometric modeling in several CAD applications still requires the notions of computations, analyses, and designs of these geometric properties. The curve modeling plays an important role in geometric modeling because it can be generalized into the development of surfaces and solids. Typically, a curve construction is based on a sequence of the given control points that approximates the shape of this curve. In other words, the specified control points influence the appearance of the curve. Besides, this curve will pass through the first and the last endpoints but does not pass through every interior point. In addition, these polynomial curves can also be differently specified according to their blending functions (polynomials), e.g., Bezier, and B-Spline curves. The models of these curves are also dissimilar from their different polynomial formulations. Classically, the presentation of Bezier curves [1-4] has been widely used in many CAD/CAM systems because they present the simplest model, defined in terms of the Bernstein basis polynomials. Among several important properties of Bezier curves are convexity, affine invariance, and Bernstein polynomial symmetry. The degree elevation and reduction [5] of Bezier curve are two promising approaches that can be efficiently applied into many consequent properties.

Some important graphics problems are two-dimensional and can be solved with curves, without the need for surface and light reflection. Examples are computer art (drawing and painting), technical drawing, and computer-aided manufacturing (CAM).

Clothes designer, for example, can benefit from a program that can draw general curves and that can later automatically cut material along a given curve.

The Bezier curves are basically and widely used in CA GD (short for Computer Aided Geometric Design). The Bezier curves were independently developed by P. de Casteljau about 1959 [6] and by P. Bezier about 1962 [3]. Bezier and de Casteljau developed their theories as part of CAD systems that were being built up at two French car companies, Renault and Citroen. Bezier curves and surfaces are now established as the mathematical basis of many CAD systems, they have also become a major tool for the development of new methods for curve and surface descriptions. The basic theory of such curves is summarized and many relevant references are also provided in [7].

Why would we want to elevate the degree of a Bezier curve? (1) In general, it is not possible to display the characteristics of a curve of degree \( n \) with a curve of degree \( n \). For example, we cannot describe a cubic curve with a quadratic function. Suppose that we are unable to produce a curve of the desired shape with a degree \( n \) Bezier curve. One option is to use a Bezier curve of higher degree.

(2) Degree elevation has important applications in surface design: for several algorithms that produce surfaces from curve input, it is necessary that these curves be of the same degree. Using degree elevation, we may achieve this by raising the degree of all the input curves to the one of the highest degree.

(3) Another application lies in the area of data transfer between different CAD/CAM or graphics systems: Suppose you have generated a parabola (i.e. a degree two Bezier curve), and you want to feed it into a system that only knows about cubics. All you have to do is degree elevate your parabola.

A lot of research [8-19] effort has gone into curves and surfaces in the last 30 years because of these reasons. Many sophisticated curve methods are known today-some are specialized and others are general purpose.

This paper is organized as follows. Section II contains the interval Bezier curves, and section III includes the basic results whereas section IV presents a numerical example, and the final section offers conclusions.

II. INTERVAL BEZIER CURVES

An interval polynomial is a polynomial whose coefficients are interval. We shall denote such polynomials in the form \( P^i(\omega) \) to distinguish them from ordinary (single-valued) polynomials. In general we express an interval polynomial of degree \( n \) in the form:

\[
P^i(\omega) = \sum_{k=0}^{n} \left[ a_k^u a_k^s \right] B_k^i(\omega), \quad \text{for all } u \in [0,1] \quad (1)
\]

in terms of the Bernstein polynomial basis.
\[ B^n_k(u) = \binom{n}{k}(1 - u)^{(n-k)}u^k, \quad \text{for } k = 0,1,\ldots,n \quad (2) \]
on $[0,1]$. Usual interval arithmetic can be applied to the interval polynomials [20].

We will define a vector-valued interval \( P^l \) in the most general terms as any compact set of points \((x,y)\) in two dimensions. The addition of such sets is given by the Minkowski sum:
\[ P^1_{x} + P^2_{y} = \{(x_{1}+x_{2},y_{1}+y_{2})|(x_{1},y_{1}) \in P^1_{x},(x_{2},y_{2}) \in P^2_{y}\} \quad (3) \]

It is prudent to restrict our attention to the vector-valued intervals that are just the tensor products of scalar intervals:
\[ P^l = [a^-,a^+] \times [b^-,b^+] = \{(x,y) | x \in [a^-,a^+] \text{ and } y \in [b^-,b^+]\} \quad (4) \]

We occasionally use the notation \(([a^-,a^+],[b^-,b^+])\) instead of \(((a^-,a^+) \times [b^-,b^+]\)) for \( P^l \). Such vector-valued intervals are rectangular regions in the plane, and their addition a trivial matter:
\[ P^1_{x} + P^2_{y} = [a^-,a^+ + c^{-},a^+ + c^{+}] \times [b^-,b^+ + d^{-},b^+ + d^{+}] \quad (5) \]

where, \( P^1_{x} = [a^-,a^+] \times [b^-,b^+] \) and \( P^2_{y} = [c^{-},c^{+}] \times [d^{-},d^{+}] \). The extension of these ideas to vector-valued intervals in spaces of higher dimension is straightforward.

An interval Bezier curve is written in the form:
\[ P^l(u) = \sum_{k=0}^{n} [p_{k}^- p_{k}^+] B^n_k(u) \quad (6) \]

where, \([p_{k}^- p_{k}^+]\) for \((k = 0,1,\ldots,n)\) are interval control points (rectangular intervals of the form (4)). For each \( u \in [0,1] \), the value \( P^l(u) \) of the interval Bezier curve (6) is a vector interval that has the following significance: For any fixed Bezier curve \( P(u) \) whose control points satisfy \( p_k \in [p_{k}^- p_{k}^+]\) for \((k = 0,1,\ldots,n)\) we have \( P(u) \in P^l(u) \). Likewise, the entire interval Bezier curve (6) defines a region in the plane that contains all Bezier curves whose control points satisfy \( p_k \in [p_{k}^- p_{k}^+]\) for \((k = 0,1,\ldots,n)\).

III. THE BASIC RESULTS

Let \( \{[p_{k}^- p_{k}^+]\}_{k=0}^{n} \) be a given set of interval control points which defines the interval Bezier curve:
\[ P^l_{k}(u) = \sum_{i=0}^{n} [p_{k}^- p_{k}^+] B^n_k(u), \quad 0 \leq u \leq 1 \quad (7) \]
of degree \( n \) where,
\[ B^n_k(u) = \binom{n}{k}(1 - u)^{(n-k)}u^k, \quad \text{for } k = 0,1,\ldots,n \quad (8) \]

The problem is to write the given interval Bezier curve in basis of degree \( n \) into basis of degree \( n+k \) without changing the curve. The new interval vertices \( \{[p_{k}^- p_{k}^+]\}_{i=0}^{n+k} \) of the new interval polygon can be found in the following way:

The four fixed Kharitonov's polynomials (four fixed Bezier curves) [21] associated with the original interval Bezier curve are:
\[
\begin{bmatrix}
\alpha_{0,n+1}^i \\
\alpha_{1,n+1}^i \\
\alpha_{2,n+1}^i \\
\vdots \\
\alpha_{n,n+1}^i \\
\alpha_{n+1,n+1}^i
\end{bmatrix} = [M_{n+1}^i]^{-1} \cdot [M_i^n] \cdot \begin{bmatrix}
\alpha_{0,n}^i \\
\alpha_{1,n}^i \\
\alpha_{2,n}^i \\
\vdots \\
\alpha_{n,n}^i \\
\alpha_{n+1,n}^i
\end{bmatrix}
\]
(12)

The condition for \( P_n^i(u) \) and \( P_{n+1}^i(u) \) to be identically equal is:

\[
\sum_{j=0}^{n+1} \binom{n+1}{j} a_j^i + \frac{\binom{n+1}{j-1}}{n+1} a_{j-1,n}^i - \frac{\binom{n+1}{j}}{n+1} a_{j+1,n}^i (1-u)^{n+1-i} t^j = 0
\]
(11)

which gives:

\[
\sum_{j=0}^{n+1} \binom{n}{j} a_j^i + \frac{\binom{n}{j-1}}{} a_{j-1,n}^i - \frac{\binom{n}{j}}{} a_{j+1,n}^i (1-u)^{n+1-i} t^j = 0
\]

or

\[
\binom{n}{j} a_j^i + \frac{\binom{n}{j-1}}{} a_{j-1,n}^i - \frac{\binom{n}{j}}{} a_{j+1,n}^i = 0
\]
(18)

This equation can be rewritten as:

\[
a_j^i + \frac{\binom{n}{j-1}}{} a_{j-1,n}^i - \frac{\binom{n}{j}}{} a_{j+1,n}^i = \frac{1}{n+1-j} \left[a_{j-1,n}^i + (n-j+1)a_{j,n}^i\right]
\]
(19)

for \( j = 0, 1, \ldots, n+1 \) and \( i = 1, 2, 3, 4 \).

Here \( a_{j-1,n}^i \) and \( a_{j+1,n}^i \) are undefined, but their coefficients are 0, so they are irrelevant in calculating \( a_{j,n+1}^i \).

Next, in the case of changing from an expression in terms of \( a_j^i \) to one in terms of \( a_j^{i+k} \), let us focus on the expression for \( a_{1,n}^i \). Using equation (19) gives:

\[
\begin{align*}
\alpha_{1,n+1}^i &= \frac{1}{n+1} \left[ a_{0,n+1}^i + na_{1,n+1}^i \right] \\
\alpha_{1,n+2}^i &= \frac{1}{n+2} \left[ a_{0,n+1}^i + (n+1)a_{1,n+1}^i \right] \\
&= \frac{1}{n+2} \left[ a_{0,n+1}^i + (n+1) \frac{1}{n+1} \left( a_{0,n}^i + na_{1,n}^i \right) \right] \\
&= \frac{1}{n+2} \left[ 2a_{0,n}^i + na_{1,n}^i \right]
\end{align*}
\]
(20)

Continuing this same procedure gives:

\[
\alpha_{1,n+k}^i = \frac{1}{n+k} \left( ka_{0,n}^i + na_{1,n}^i \right) \text{ for } i = 1, 2, 3, 4
\]
(21)

where,

\[
a_{0,n+k}^i = a_{0,n}^i \text{ for } i = 1, 2, 3, 4
\]
(22)

Finally, the new interval vertices of the new interval polygon can be obtained as follows:

\[
\left[ \beta_j^- \beta_j^+ \right] = [\min(a_{j,n+k}^i), \max(a_{j,n+k}^i)]
\]
(23)

\( j = 0, 1, \ldots, n+k \) and \( i = 1, 2, 3, 4 \)

IV. NUMERICAL EXAMPLE

Consider the quadratic interval Bezier curve defined by the three interval control points:

\[
\begin{align*}
\beta_0^- & = (1.6000, 1.8500) \\
\beta_0^+ & = (1.4000, 1.7500) \\
\beta_1^- & = (2.8000, 3.2000) \\
\beta_1^+ & = (4.2000, 4.6500) \\
\beta_2^- & = (6.2500, 6.7500) \\
\beta_2^+ & = (1.8000, 2.2500)
\end{align*}
\]
(24)

For all \( u \) in the range \( 0 \leq u \leq 1 \) we must have:

\[
P_n^i(u) = P_{n+1}^i(u) \text{ for } (i = 1, 2, 3, 4)
\]
(25)
It is required to elevate the degree of the given interval Bezier curve defined by them to 3 without changing its shape.

The Bezier matrices $M^i_2$ and $M^i_4$ are obtained as follows:

\[
M^i_2 = \begin{bmatrix}
0 & 0 & 0 \\
1 & -2 & 1 \\
-2 & 2 & 0 \\
1 & 0 & 0
\end{bmatrix} \quad \text{and} \quad M^i_4 = \begin{bmatrix}
-1 & 3 & -3 & 1 \\
3 & -6 & 3 & 0 \\
-3 & 3 & 0 & 0 \\
1 & 0 & 0 & 0
\end{bmatrix} 
\]

\((i = 1, 2, 3, 4)\)

The new interval vertices $\left\{ [\beta^*_i, \beta^*_i] \right\}_{i=0}^{2}$ of the new interval polygon are obtained as explained in section III:

\[
\begin{align*}
[\beta^*_i, \beta^*_i] &= \left\{ [1.6000, 1.8500], [1.4000, 1.7500] \right\} \\
[\beta^*_i, \beta^*_i] &= \left\{ [2.4000, 2.7500], [3.2667, 3.6833] \right\} \\
[\beta^*_i, \beta^*_i] &= \left\{ [3.9500, 4.3833], [3.4000, 3.8500] \right\} \\
[\beta^*_i, \beta^*_i] &= \left\{ [6.2500, 6.7500], [1.8000, 2.2500] \right\}
\end{align*}
\]

Simulation results in Figure (1) shows the envelopes of the original interval Bezier curve and the elevated interval Bezier curve, respectively.

V. CONCLUSIONS

A simple matrix form for degree elevation of interval Bezier curve is presented in this paper. The four fixed Kharitonov's polynomials (four fixed Bezier curves) associated with the original interval Bezier curve are obtained. The method is based on the matrix representations of the degree elevation process. The process of degree elevations $k$ times are applied to the four fixed Bezier curves of degree $n$ to obtain the four fixed Bezier curves of degree $n + k$ without changing their shapes. Finally the new interval vertices $\left\{ [\beta^*_i, \beta^*_i] \right\}_{i=0}^{2+k}$ of the new interval polygon are obtained from vertices of the new fixed polygons of the four fixed Bezier curves. In general, it is not possible to display the characteristics of a curve of degree $n + 1$ with a curve of degree $n$. Degree elevation has important applications in surface design. Another application lies in the area of data transfer between different CAD/CAM or graphics systems.