Calculating of Natural Frequency of Stepping Cantilever Beam

Dr. Luay S. Al-Ansari

Ls_alansari@yahoo.com
University of Kufa / Faculty of Mechanical Engineering

***Abstract*--** **In this work, three models are used to calculate the natural frequency of cantilever stepping beam compound from two parts. These models are Rayleigh model, modified Rayleigh model, and Finite elements model (ANSYS model). The Rayleigh model is modified by calculating the stiffness at each point of the beam. The modified Rayleigh model is much closer to the ANSYS model than the Rayleigh model. The comparison between the three methods was presented in this study and the convergence for the three methods of study was shown. The results showed the effect of the width for small and large part of beam, the length of large part of step, and the ratio of large to small width of stepping beam on the natural frequency of stepping beam. The natural frequency of stepping beam is increasing with increasing of the width of small and large parts of beam. In addition to, the natural frequency of beam is increasing with increasing the length of large width until reach to (0.52 m) and decreasing then when the modified Rayleigh model or ANSYS model are used.**

***Index Term*--** **Natural Frequency, Stepping Cantilever Beam, Rayleigh Method, Modified Rayleigh Method, Finite Elements Method, ANSYS, Stiffness of Beam, Equivalent Moment of Inertia, Point Equivalent Moment of Inertia.**

1. Introduction

Beams with variable cross-section and/or material properties are frequently used in aeronautical engineering (e.g., rotor shafts and functionally graded beams), mechanical engineering (e.g., robot arms and crane booms), and civil engineering (e.g., beams, columns, and steel composite floor slabs in the single direction loading case).

The beam with variable cross-section is often modeled by a large number of small uniform elements, replacing the continuous changes with a step law. This scheme is accurate for a stepped beam but approximated for a beam with continuously changed cross-section. Although in this way, it is always possible to reduce errors as much as desired and obtain acceptable results by refining meshes, the modeling and computational efforts can become excessive. The boundary element methods for static torsion and torsional vibration analyses of bars with variable cross-section were developed by Sapountzakis and coworkers [1, 2]. The dynamic stiffness method to investigate the free bending vibration of rotating beams with linearly changed cross-section was used by Banerjee et al. [3].The static responses of curved beam with variable cross-section was studied [4], in which the stiffness matrix and the equivalent nodal loads of the curved beam element were presented. The Carrera Unified Formulation was derived by Carrera and coworkers [5, 6], and under that framework, they presented a method to analyze beams with arbitrary cross-sectional geometries. Firouz-Abadi et al. [7] presented a Wentzel, Kramers, Brillouin approximation-based analytical solution to free transverse vibration of a class of varied cross-section beams. Ece et al. [8] solved the problem of the vibration of beams with exponentially varying cross-section width for three different types of boundary conditions associated with simply supported, clamped, and free ends. Shin et al. [9] applied the generalized differential quadrature method and differential transformation method to vibration analysis of circular arches with variable cross-section, stating that these two methods showed fast convergence and accuracy. Exact displacement interpolation functions to beams with linearly changed cross-section were solved and then used to derive the accurate stiffness matrix [10, 11]. The use of exact displacement interpolation functions to solve varied cross-section beam problems is a straightforward way; however, they [10, 11] only focused on the beam with linearly and continuously changed cross-section. A special case is the stepped beam, a beam with abrupt changes of cross-section and/or material properties. Several works about stepped beams had been published. An analytical approach to calculating the frequencies of beams on elastic end supports and with up to three step changes in cross-section was presented by Naguleswaran [12]. Maurini et al. [13] presented an enhancement of assumed mode method by introducing special jump functions to catch the curvature discontinuities of the mode shapes. Kisa and Gurel [14] presented a technique to solve the free vibration problems of stepped beam with circular cross-section and an existing crack. In [15, 16], solutions to the free vibration problem of stepped beams were presented by using the properties of Green's function. Jaworski and Dowell [17] conducted an experiment of free vibration analysis of a stepped cantilevered beam and compared the experiment results with the classical Rayleigh-Ritz method, component modal analysis, and commercial finite element software ANSYS, the local boundary conditions and non beam effects were discussed. The composite element method to analyze free and forced vibrations of stepped beams was used Lu et al.[18] and the theoretical results compared with experimental results. Mao and Pietrzko [19] used the Adomian decomposition method to investigate the free vibrations of a two-stepped beam, considering different boundary conditions, step locations, and step ratios.Most recently, Zheng and Ji [20] presented an equivalent representation of a stepped beam with a uniform beam to simplify the calculation of static deformations and frequencies. The methods to analyze the stepped beams and the beams with continuously changed cross-section are generally not unified.

In addition to cross-sectional geometry, other parameters such as material properties (e.g., modulus, mass density, etc.) are also changeable, which belong to the researches on beams made of functionally graded materials. For functionally graded beams, continuous material gradient

variation may be orientated in the cross-section and/or in the axial direction. For the former, there have been many researches devoted to static and vibration analyses (e.g.,[21–25]). For the axially functionally graded beams, few solutions are yet found for arbitrary gradient changes [26, 27].

2. Rayleigh Method

Rayleigh method is a good method and simpler than the other numerical methods for finding the natural frequencies. It includes calculating the kinetic energy and potential energy of the system. The kinetic energy can be calculated by integrating the mass through length of the beam and the potential energy by integrating the stiffness through the length of the beam. So one can get [28, , 29, 17]:

$$\omega^2 = \frac{\int_0^l EI\left(\frac{d^2y(x)}{dx^2}\right)^2 dx}{\int_0^l \rho A(y(x))^2 dx} = \frac{g\sum_{i=1}^{n+1} m_i y_i}{\sum_{i=1}^{n+1} m_i y_i^2} \qquad (1)$$

Where: (ω) is frequency, (E) is Modulus of Elasticity,(I) is Moment of Inertia, (ρ) is Density, (A) is Cross Section Area, (m) mass, and (y) is Deflection.

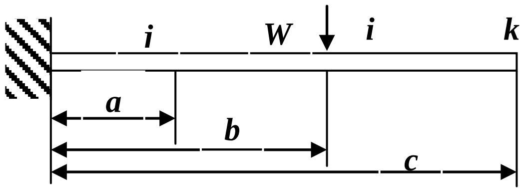
By calculating the deflection of the beam(y(x)) using the following steps:

(1) Dividing the beam into (n) parts (i.e. (n+1) nodes).
(2) Calculate the delta matrix $[\delta]_{((n+1)*(n+1))}$ using Table (1).
(3) Calculate the mass matrix $[m]_{((n+1))}$.
(4) Calculate the deflection at each node by multiplying delta matrix and mass matrix ($[y]_{(n+1)} = [\delta]_{((n+1)*(n+1))}[m]_{((n+1))}$) after applying the boundary conditions.

A MATLAB program was used to simulate the Rayleigh method in order to calculate the first natural frequency of any beam (Different materials, different dimensions and different cross section area).

TABLE I
Formulae of the Deflections of the Cantilever Beams[28].

$$\delta_{ji} = \frac{Wa^2(3b-a)}{6EI},\ \delta_{ii} = \frac{Wb^3}{3EI},\ \delta_{ki} = \frac{Wb^2(3c-b)}{6EI}$$



III. Disadvantage of Rayleigh Method

From equation (1) and Table (1), one of the most important parameter affects on the value of deflection is the moment of inertia. In Rayleigh method, the moment of inertia was calculated using the equivalent inertia technique.

In order to explain the equivalent inertia technique, consider a cantilever stepped beam with two steps as shown in Fig. (1). Since the dimensions of one step differ from the other, therefore the moment of inertia of the first part differ than that of the second part in additional to the difference in the cross section area.

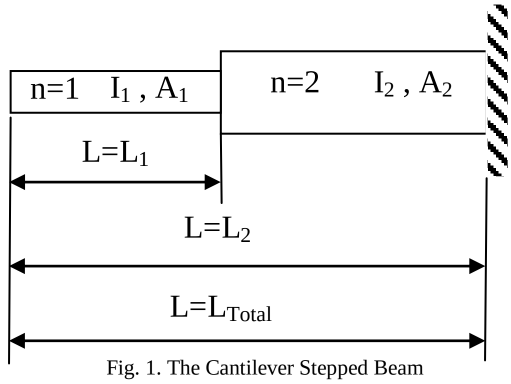


Fig. 1. The Cantilever Stepped Beam

The equivalent moment of inertia can be calculated by the following equation:

$$I_{eq} = \frac{(L_{Total})^3}{\left[\frac{(L_1)^3}{I_1} + \frac{(L_2)^3-(L_1)^3}{I_2}\right]} = \frac{(L_{Total})^3}{\sum_{n=1}^{2}\left[\frac{(L_n)^3-(L_{n-1})^3}{I_n}\right]} \qquad (2)$$

and for Nth stepped beam equation (2) can be written as:

$$I_{eq} = \frac{(L_{Total})^3}{\sum_{n=1}^{N}\left[\frac{(L_n)^3-(L_{n-1})^3}{I_n}\right]} \qquad (3)$$

This technique is suitable for small value of (I_2/I_1) ratio and when this ratio is large the equivalent moment of inertia technique falls to calculate the natural frequency.

The Point Equivalent Moment of Inertia:

In order to modify the Rayleigh method, the point equivalent moment of inertia must be used in order calculate the natural frequency. The idea of point equivalent moment of inertia is concentrated on equation (2) and equation (3). In these equations, the parameters, affect on the value of equivalent moment of inertia, are the length of steps and the dimensions of cross section area of the steps. The length of small part can be expressed as a function of distance along the beam. Now if the beam is only large part then the equivalent moment of inertia will be (I_2). But if the beam consists of two steps, the equivalent moments of inertia at the points that lie on the large step are (I_2). While the equivalent moment of inertia at the points that lie on the small step are changed and depend on the position of step change. The equivalent moment of inertia at any point lies on the small step can be written as:

$$I_{eq} = \frac{(L_{Total})^3}{\left[\frac{(L_1(x))^3}{I_1} + \frac{(L_2)^3-(L_1(x))^3}{I_2}\right]} \qquad (4)$$

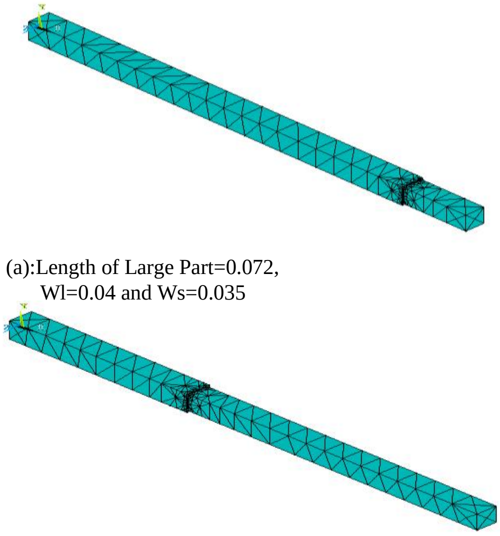
V. The Modified Rayleigh Method

For modifying the Rayleigh method, the point equivalent moment of inertia was used instead the classical equivalent moment of inertia to calculate the natural frequency of stepped cantilever beam. A new MATLAB program was used to simulate the modified Rayleigh method to calculate the first natural frequency of any stepped beam of two steps [30].

VI. Numerical Approach (Finite Elements Method)

In this method, the finite elements method was applied by using the ANSYS program. The three dimensional model were built and the element (Solid Tet 10 node 187) were used [31,17]. Generally the number of nodes was

approximately (1100-2500) and the number of elements was (400-1500). A sample of meshed beam is shown in Fig.(2).



(a):Length of Large Part=0.072, Wl=0.04 and Ws=0.035

(b): Length of Large Part=0.032 Wl=0.04 and Ws=0.035

Fig. 2. Sample of Meshed Stepping beam.

VII. RESULTS AND DISCUSSION

Fig.(3) shows the stepped beam that was studied in this work. This beam is square cross section area.

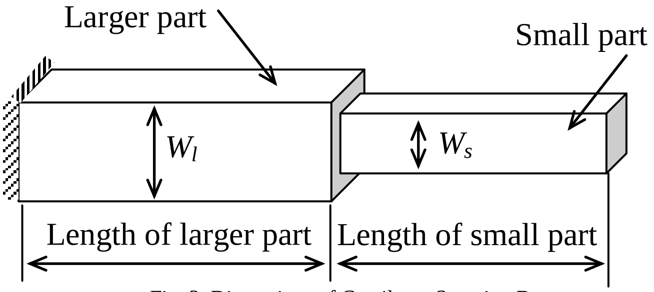


Fig. 3. Dimensions of Cantilever Stepping Beam

The cases that were studied in this work, can be listed in Table (2) and the material properties of the beam were (Modulus of Elasticity = 200 GPa.; Poisson Ratio = 0.3 and Density = 7860 kg/m^3) and the length of the beam was (84) cm.

1- The point Equivalent Moment of Inertia

The stiffness of the beam is one of the most important parameter affecting in static and dynamic analysis. The stiffness in vibration means the multiplying modulus of elasticity and moment of inertia of beam. Therefore, the stiffness was studied in this work instead of moment of inertia.

Fig.(4) shows the varying of stiffness at any point in the beam along the length of the beam with the different length of large part when the width of large part is (0.01 m) and the width of small part is (0.005 m). Two straight lines can be distinguished in the figure. The first straight line refers to the stiffness of the beam when the length of large part equals zero. In other words, the beam has one cross section area (0.005 m*0.005 m). The second straight line refers to the stiffness of the beam when the length of large part equals the length of the beam (i.e. the beam has one cross section area (0.01 m*0.01 m)). For any length of large part, the stiffness decreases from larger value (I_{Large}) to the smaller one (I_{Small}). When the length of large part increases the stiffness value may be not reach to smaller value. Also, the decreasing rate of stiffness decreases when the length of large part increases. The same observation notes can be seen in Fig.(5)-Fig.(9).

From figures (4-9), the deceasing rate of the stiffness decreases when the ratio between the widths of the large part to that of small part (W_L/W_S) increases.

2- The Natural Frequency:

As mentioned before, three models were made in order to calculate natural frequency of stepped beam. These models were Rayleigh model, modified Rayleigh model and finite elements model (ANSYS model). Fig. 10 and Fig. 15. show the comparison between natural frequencies of the three models when the length of large part of beam increases for width of the small part (W_{Small}) equals (0.005 m) and different width of the large.

Fig.(16)-Fig.(21) show the comparison between natural frequencies of the three models when the length of large part of beam increases for width of the small part (W_{Small}) equals (0.01 m) and different width of the large part (W_{Large}). A good agreement between the modified Rayleigh model and ANSYS model for the small value of the width of the large part (W_{Large}) can be seen and the maximum difference between modified Rayleigh model and ANSYS model still lies in (L Large =0.52 m).

Fig.(22)-Fig.(26) show the comparison between natural frequencies of the three models when the length of large part of beam increases for width of the small part (W_{Small}) equals (0.015 m) and different width of the large part (W_{Large}). A good agreement between the modified Rayleigh model and ANSYS model for the small value of the width of the large part (W_{Large}) can be seen. Also the maximum difference between modified Rayleigh model and ANSYS model still lies in (L Large =0.52 m).

The same remarks can be seen in Fig.(27)-Fig.(30) when (W_{Small}) equals to (0.02 m) , Fig.(31)-Fig.(33) when (W_{Small}) equals to (0.025 m) , Fig.(34) and Fig.(35) when (W_{Small}) equals to (0.03 m)) and in Fig.(36) when (W_{Small}) equals to (0.035 m). Generally, when the ratio (W_{large}/ W_{Small}) increases the deference between the natural frequency of modified Rayleigh model and ANSYS model will be increase too, this happened because of the effect of stiffness in each point on the beam.

TABLE II
DIMENSIONS OF THE STEPPED BEAM THAT WERE STUDIED IN THIS WORK.

No.	Width of Large Part (m).	Width of Small Part (m).	Length of Large Part (m).	Length of Small Part (m).
1.	0.01	0.005	0.12, 0.21, 0.32, 0.42, 0.52, 0.63, 0.72 respectively	0.72, 0.63, 0.52, 0.42, 0.32, 0.21, 0.12 respectively
2.	0.015	0.005	0.12, 0.21, 0.32, 0.42, 0.52, 0.63, 0.72 respectively	0.72, 0.63, 0.52, 0.42, 0.32, 0.21, 0.12 respectively
3.		0.01		
4.	0.02	0.005	0.12, 0.21, 0.32, 0.42, 0.52, 0.63, 0.72 respectively	0.72, 0.63, 0.52, 0.42, 0.32, 0.21, 0.12 respectively
5.		0.01		
6.		0.015		
7.	0.025	0.005	0.12, 0.21, 0.32, 0.42, 0.52, 0.63, 0.72 respectively	0.72, 0.63, 0.52, 0.42, 0.32, 0.21, 0.12 respectively
8.		0.01		
9.		0.015		
10.		0.02		
11.	0.03	0.005	0.12, 0.21, 0.32, 0.42, 0.52, 0.63, 0.72 respectively	0.72, 0.63, 0.52, 0.42, 0.32, 0.21, 0.12 respectively
12.		0.01		
13.		0.015		
14.		0.02		
15.		0.025		
16.	0.035	0.005	0.12, 0.21, 0.32, 0.42, 0.52, 0.63, 0.72 respectively	0.72, 0.63, 0.52, 0.42, 0.32, 0.21, 0.12 respectively
17.		0.01		
18.		0.015		
19.		0.02		
20.		0.025		
21.		0.03		
22.	0.04	0.005	0.12, 0.21, 0.32, 0.42, 0.52, 0.63, 0.72 respectively	0.72, 0.63, 0.52, 0.42, 0.32, 0.21, 0.12 respectively
23.		0.01		
24.		0.015		
25.		0.02		
26.		0.025		
27.		0.03		
28.		0.035		

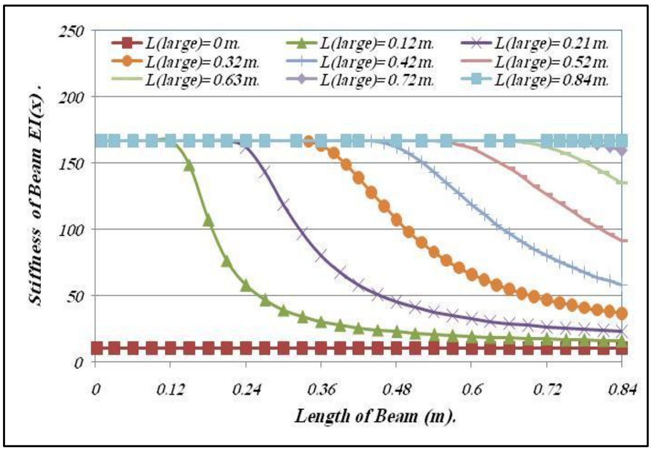


Fig. 4. Variation of Stiffness of Beam Along the Length of Beam When W_L=0.01 m and W_S=0.005 m.

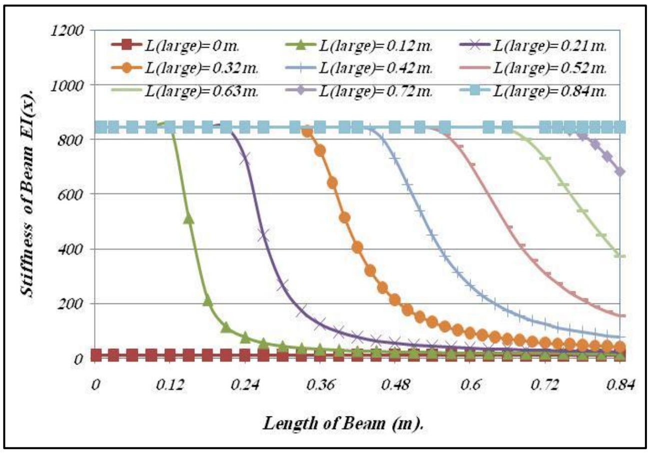


Fig. 5. Variation of Stiffness of Beam Along the Length of Beam When W_L=0.015 m and W_S=0.005 m.

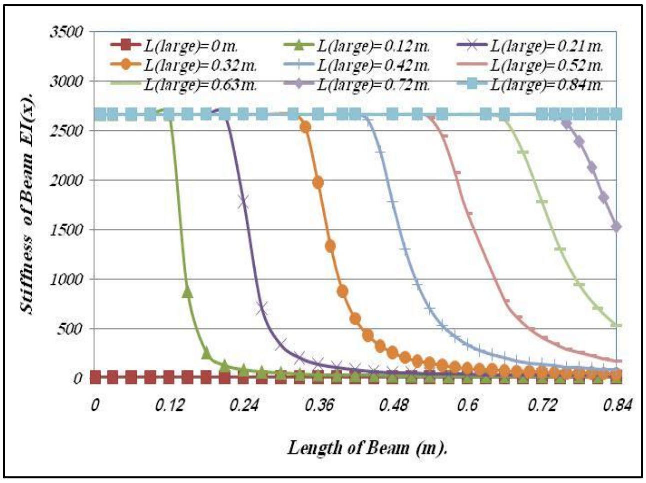


Fig. 6. Variation of Stiffness of Beam Along the Length of Beam When W_L=0.02 m and W_S=0.005 m.

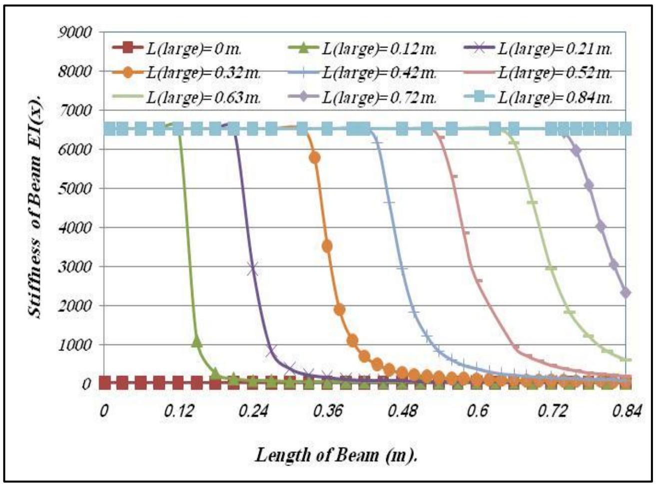


Fig. 7. Variation of Stiffness of Beam Along the Length of Beam When W_L=0.025 m and W_S=0.005 m.

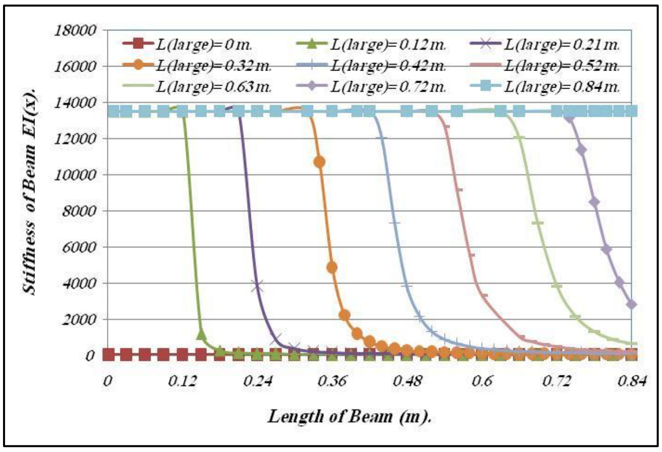


Fig. 8. Variation of Stiffness of Beam Along the Length of Beam When W_L=0.03 m and W_S=0.005 m.

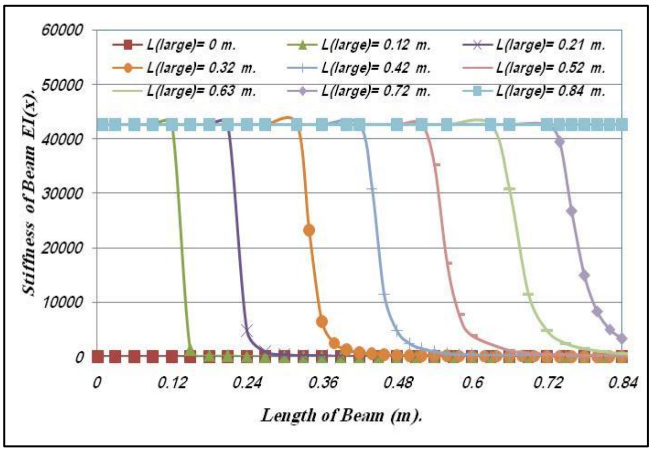


Fig. 9. Variation of Stiffness of Beam Along the Length of Beam When W_L=0.04m and W_S=0.005 m.

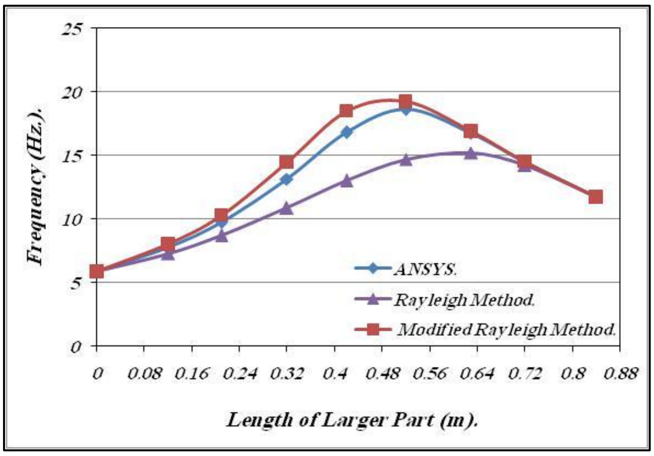


Fig. 10. Comparison Between the Natural Frequency of the Three Methods with the Length of Larger Part When (W_L=0.01m) and (W_S=0.005m).

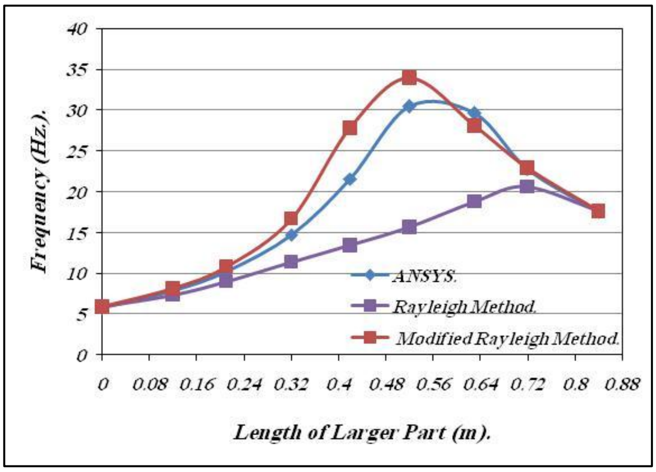


Fig. 11. Comparison Between the Natural Frequency of the Three Methods with the Length of Larger Part When (W_L=0.015m) and (W_S=0.005m).

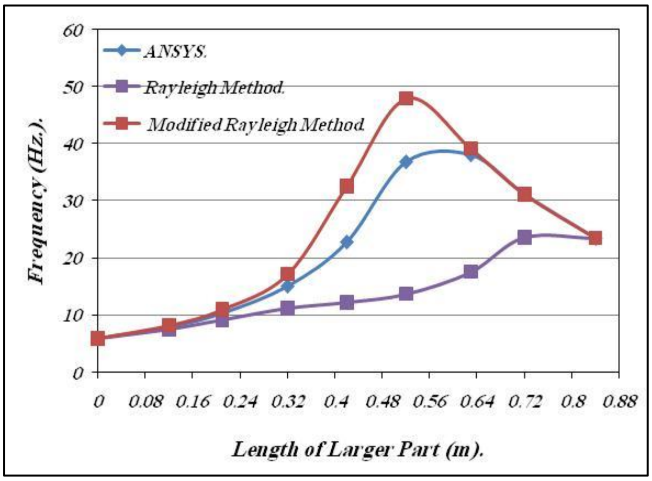


Fig. 12. Comparison Between the Natural Frequency of the Three Methods with the Length of Larger Part When (W_L=0.02m) and (W_S=0.005m).

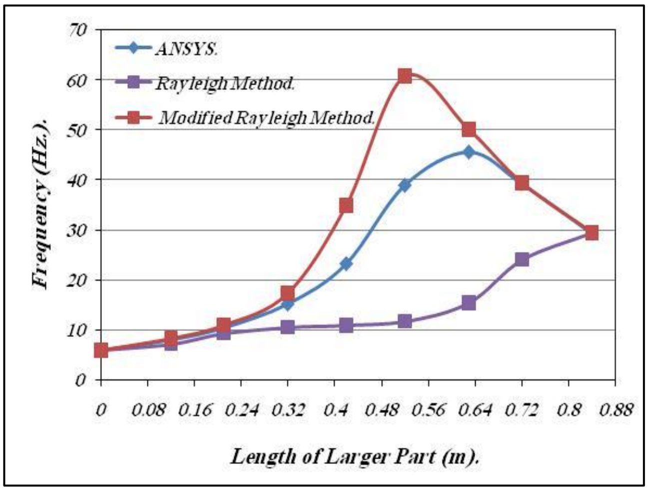


Fig. 13. Comparison Between the Natural Frequency of the Three Methods with the Length of Larger Part When (W_L=0.025m) and (W_S=0.005m).

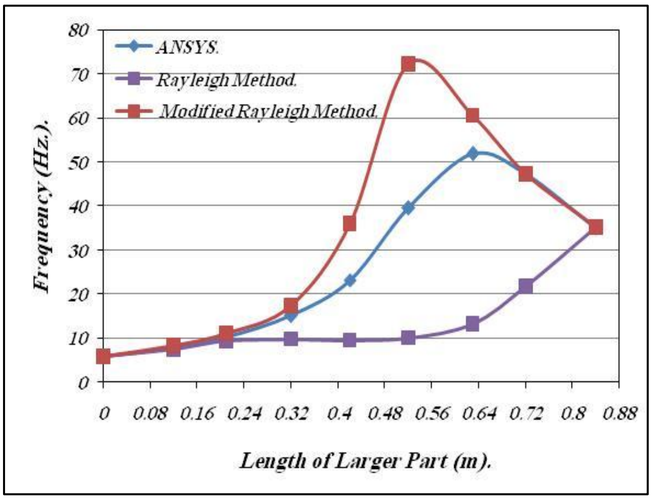


Fig. 14. Comparison Between the Natural Frequency of the Three Methods with the Length of Larger Part When (W_L=0.03m) and (W_S=0.005m).

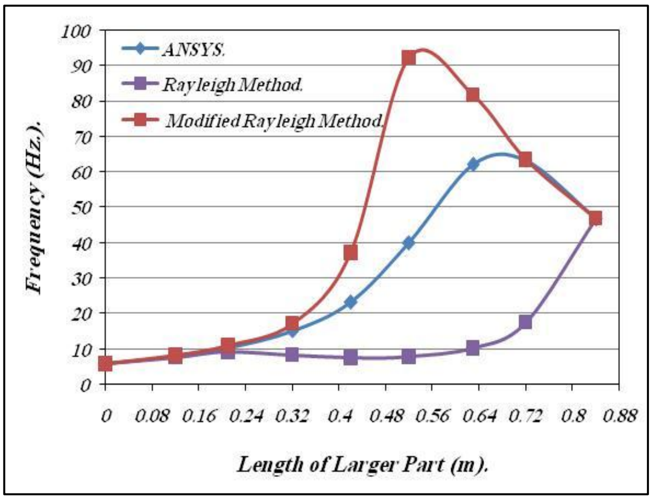


Fig. 15. Comparison Between the Natural Frequency of the Three Methods with the Length of Larger Part When (W_L=0.04m) and (W_S=0.005m).

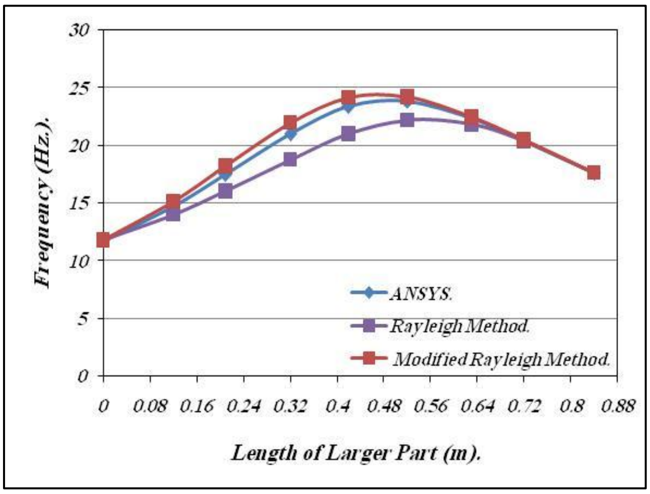


Fig. 16. Comparison Between the Natural Frequency of the Three Methods with the Length of Larger Part When (W_L=0.015m) and (W_S=0.01m).

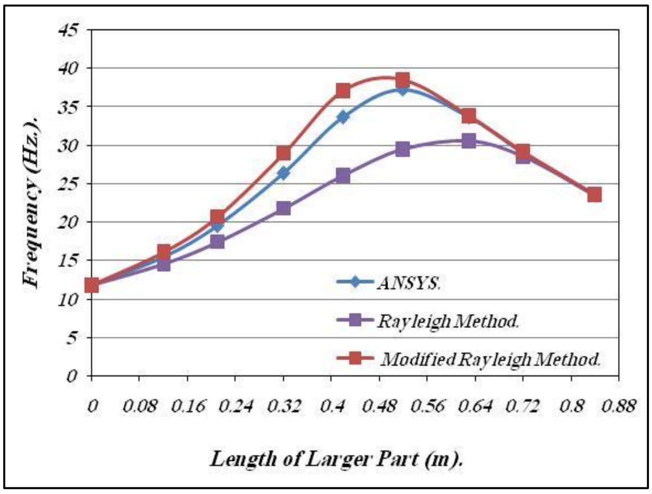


Fig. 17. Comparison Between the Natural Frequency of the Three Methods with the Length of Larger Part When (W_L=0.02m) and(W_S=0.01m).

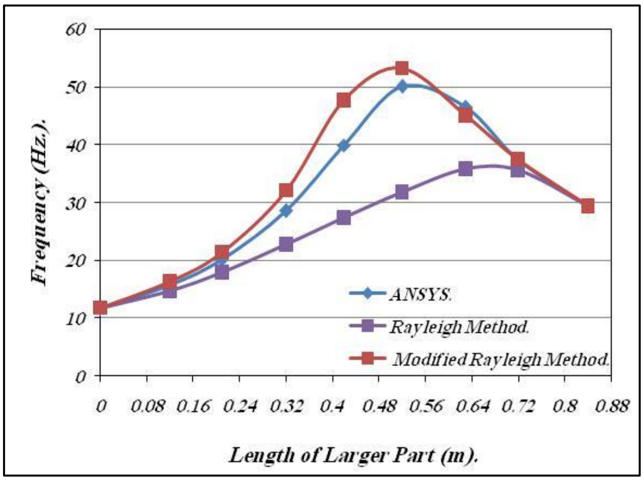


Fig. 18. Comparison Between the Natural Frequency of the Three Methods with the Length of Larger Part When (W_L=0.025m) and(W_S=0.01m).

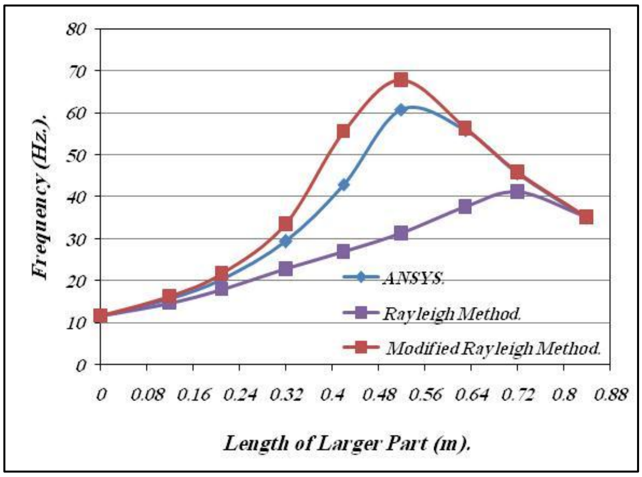


Fig. 19. Comparison Between the Natural Frequency of the Three Methods with the Length of Larger Part When (W_L=0.03m) and (W_S=0.01m).

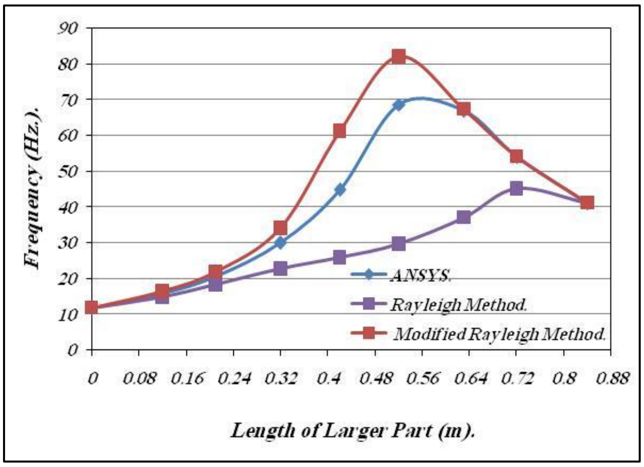


Fig. 20. Comparison Between the Natural Frequency of the Three Methods with the Length of Larger Part When (W_L=0.035m) and(W_S=0.01m).

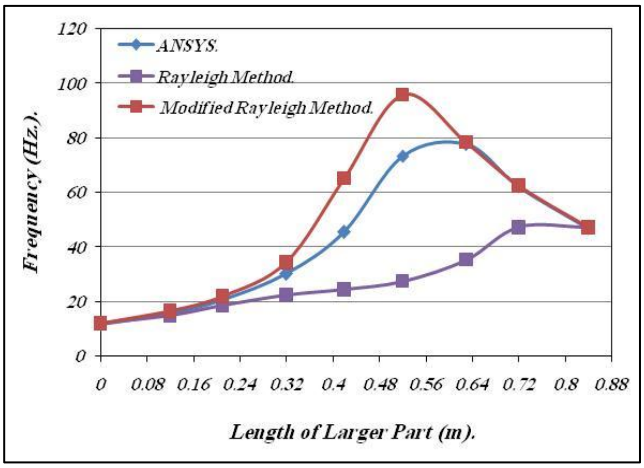


Fig. 21. Comparison Between the Natural Frequency of the Three Methods with the Length of Larger Part When (W_L=0.04m) and (W_S=0.01m).

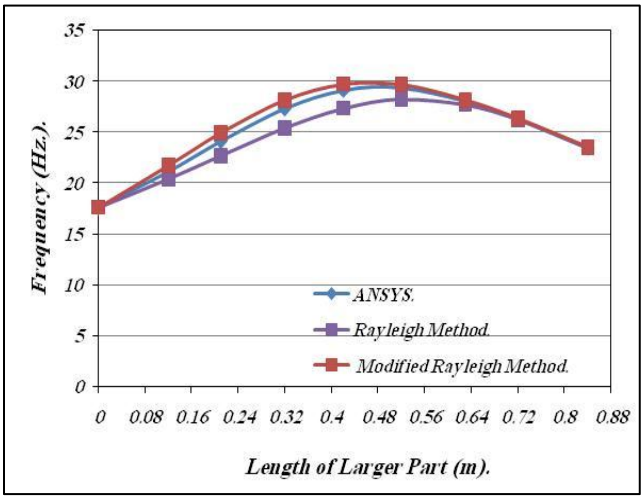


Fig. 22. Comparison Between the Natural Frequency of the Three Methods with the Length of Larger Part When (W_L=0.02m) and (W_S=0.015m).

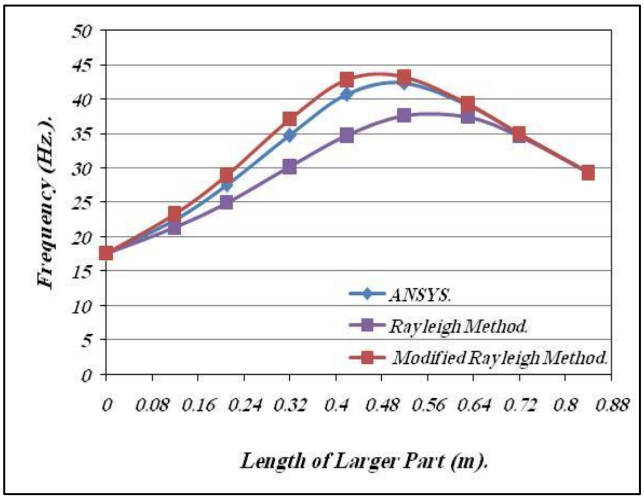


Fig. 23. Comparison Between the Natural Frequency of the Three Methods with the Length of Larger Part When (W_L=0.025m) and (W_S=0.015m).

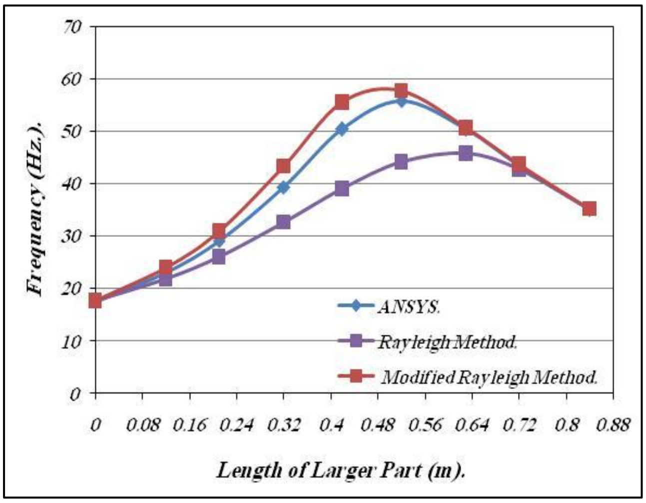


Fig. 24. Comparison Between the Natural Frequency of the Three Methods with the Length of Larger Part When (W_L=0.03m) and (W_S=0.015m).

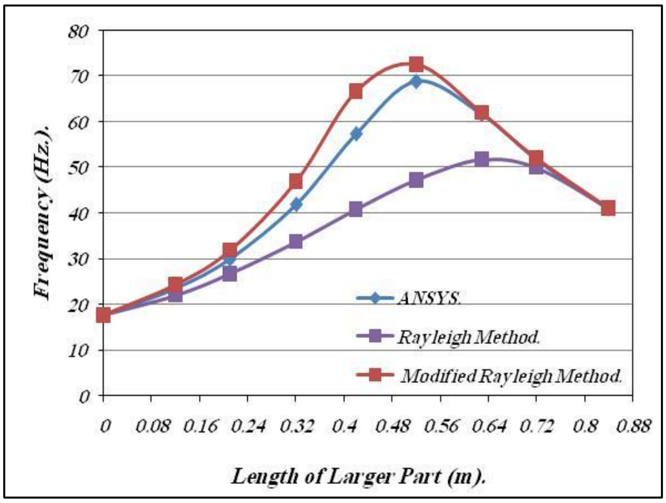


Fig. 25. Comparison Between the Natural Frequency of the Three Methods with the Length of Larger Part When (W_L=0.035m) and (W_S=0.015m).

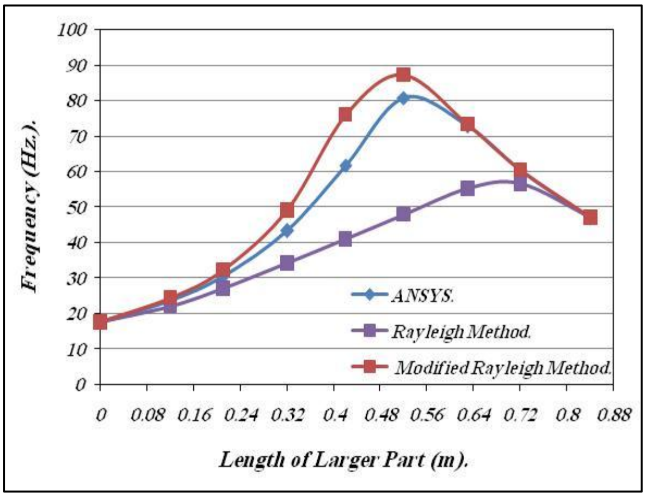


Fig. 26. Comparison Between the Natural Frequency of the Three Methods with the Length of Larger Part When (W_L=0.04m) and (W_S=0.015m).

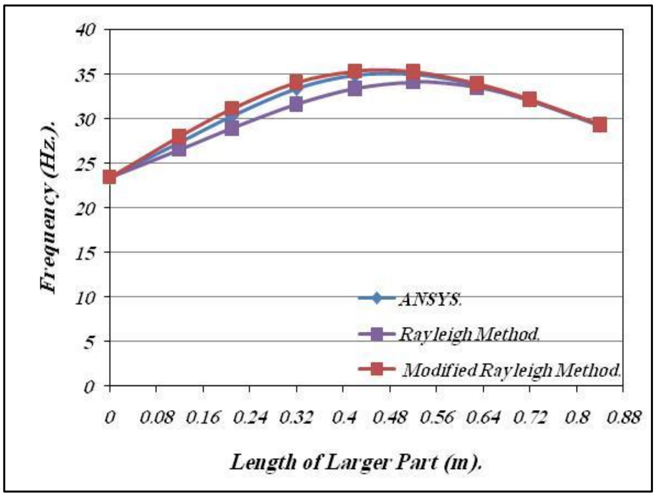


Fig. 27. Comparison Between the Natural Frequency of the Three Methods with the Length of Larger Part When (W_L=0.025m) and (W_S=0.02m).

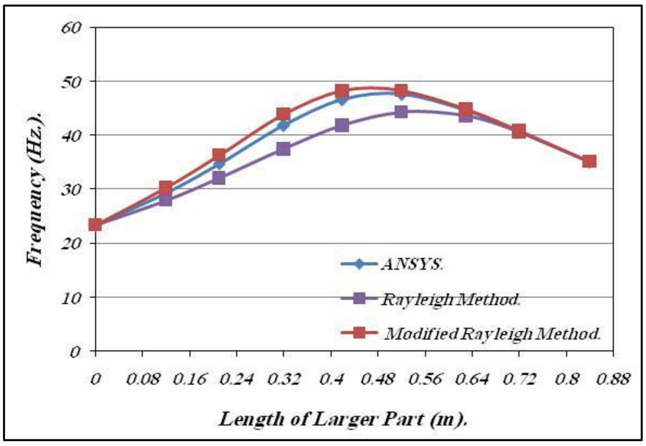


Fig. 28. Comparison Between the Natural Frequency of the Three Methods with the Length of Larger Part When (W_L=0.03m) and (W_S=0.02m).

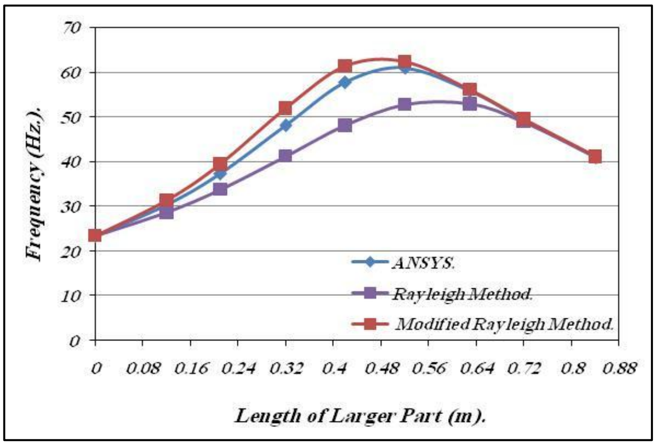


Fig. 29. Comparison Between the Natural Frequency of the Three Methods with the Length of Larger Part When (W_L=0.035m) and (W_S=0.02m).

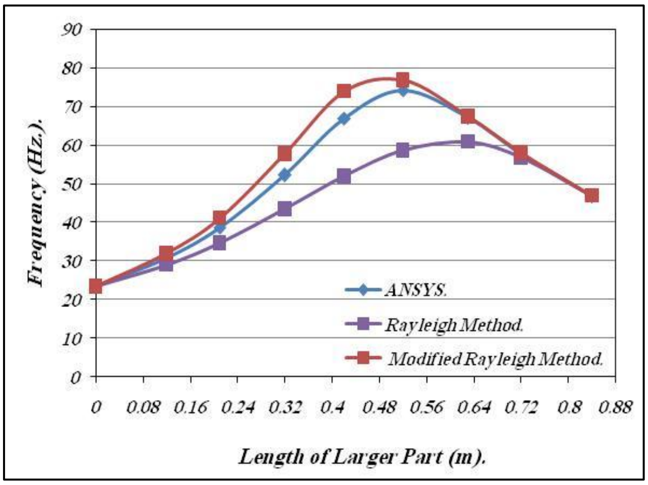


Fig. 30. Comparison Between the Natural Frequency of the Three Methods with the Length of Larger Part When (W_L=0.04m) and (W_S=0.02m).

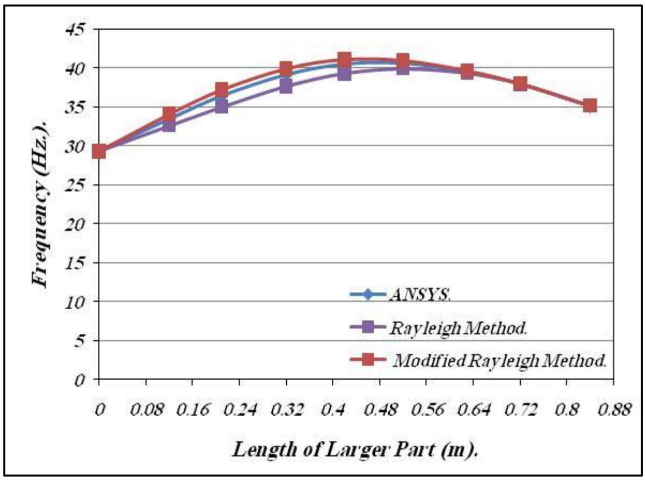


Fig. 31. Comparison Between the Natural Frequency of the Three Methods with the Length of Larger Part When (W_L=0.03m) and (W_S=0.025m).

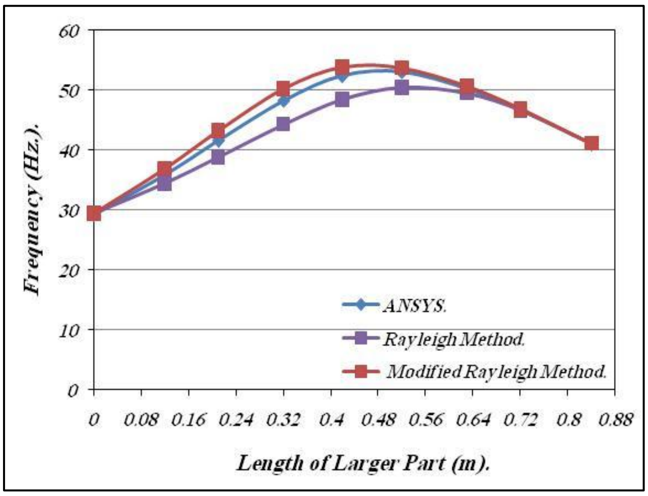


Fig. 32. Comparison Between the Natural Frequency of the Three Methods with the Length of Larger Part When (W_L=0.035m) and (W_S=0.025m).

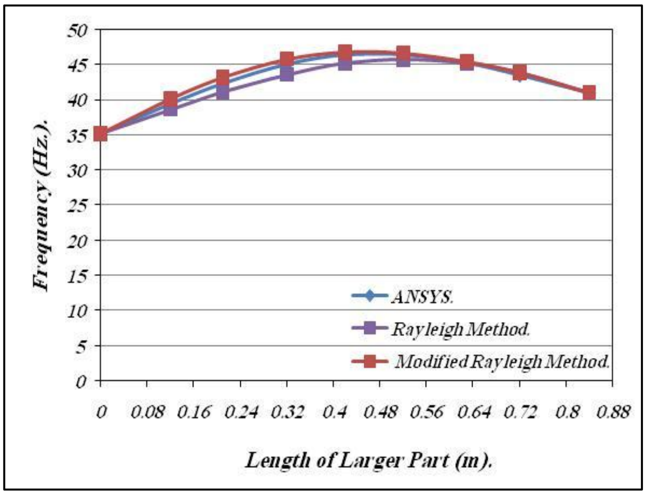


Fig. 34. Comparison Between the Natural Frequency of the Three Methods with the Length of Larger Part When (W_L=0.035m) and (W_S=0.03m).

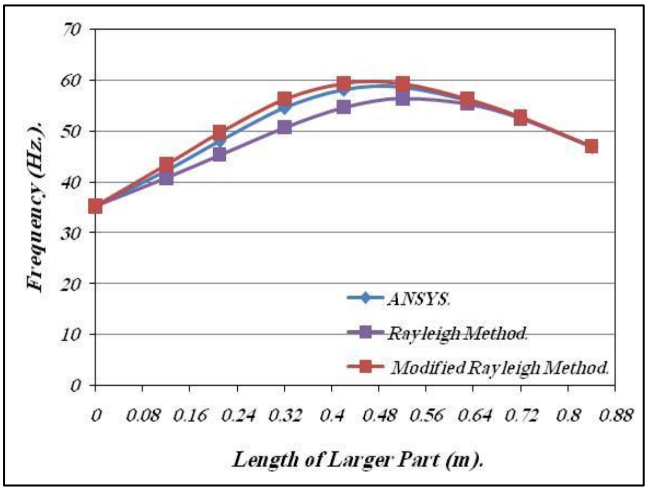


Fig. 35. Comparison Between the Natural Frequency of the Three Methods with the Length of Larger Part When (W_L=0.04m) and (W_S=0.03m).

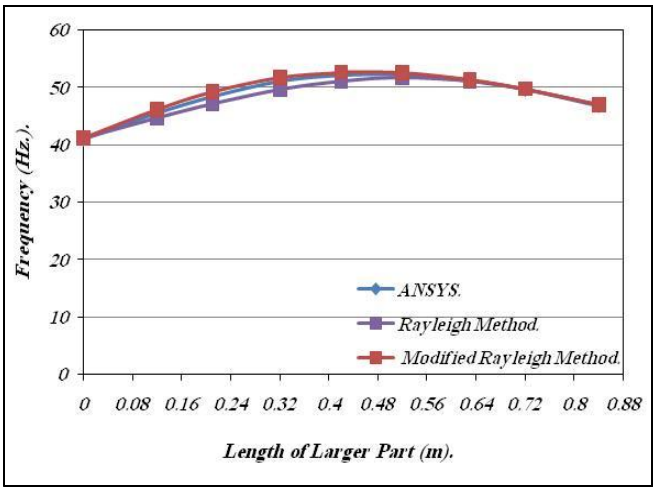


Fig. 36. Comparison Between the Natural Frequency of the Three Methods with the Length of Larger Part When (W_L=0.04m) and (W_S=0.035m).

VIII. CONCLUSION

Generally, the stiffness of stepped beam depends on length and width of large part of beam and length and width of small part of beam in additional to the modulus of elasticity of beam material. In this work and according to the point equivalent moment of inertia, the stiffness of any point at the beam depends

on length and width of large part of beam, length and width of small part of beam and the modulus of elasticity of beam material in additional to the position of the point. When the ratio (I_{Large} / I_{small}) or ($Area_{Large} / Area_{small}$) (or ($W_{Large} / W_{small}$)) increases, the decreasing rate of stiffness will be increases.

The natural frequency of any stepped beam depends on the parameters affects on the stiffness of beam when the density is constant. If the length of large part of beam increases, the natural frequency will be increase until reach to (0.52 m) and then will be decrease for any value of width of small part when the modified Rayleigh model was used. While the natural frequency will be increase until reach to region (0.042-0.63) m and then will be decrease for any value of width of small part when the ANSYS model was used.

For this kind of stepped beam, the modified Rayleigh model is a suitable method for calculating the natural frequency especially when the ratio (W_{Large} / W_{small}) is small. The modified Rayleigh model is much closer to the ANSYS model than the Rayleigh model.

For future works, the change in stiffness of stepping beam due to crack can be calculating. Also, the effect of dimensions of cross section areas which is stepping on the stiffness of beam can be studied. In other hand, the number of steps will affected on the stiffness of beam.

REFERENCES

[1] E. J. Sapountzakis and V. G.Mokos, “Nonuniform torsion of bars of variable cross section,” Computers and Structures, vol. 82, no. 9-10, pp. 703–715, 2004.

[2] E. J. Sapountzakis, “Torsional vibrations of composite bars of variable cross-section by BEM,” Computer Methods in Applied Mechanics and Engineering, vol. 194, no. 18–20, pp. 2127–2145, 2005.

[3] J. R. Banerjee, H. Su, and D. R. Jackson, “Free vibration of rotating tapered beams using the dynamic stiffness method,” Journal of Sound and Vibration, vol. 298, no. 4-5, pp. 1034–1054, 2006.

[4] F. N. Gimena, P. Gonzaga, and L. Gimena, “3D-curved beam element with varying cross-sectional area under generalized loads,” Engineering Structures, vol. 30, no. 2, pp. 404–411, 2008.

[5] E. Carrera, G. Giunta, and M. Petrolo, Beam Structures: Classical and Advanced Theories, John Wiley & Sons, Chichester, UK, 2011.

[6] E. Carrera, G. Giunta, P. Nali, and M. Petrolo, “Refined beam elements with arbitrary cross-section geometries,” Computers and Structures, vol. 88, no. 5-6, pp. 283–293, 2010.

[7] R. D. Firouz-Abadi, H. Haddadpour, and A. B. Novinzadeh, “An asymptotic solution to transverse free vibrations of variable-section beams,” Journal of Sound and Vibration, vol. 304, no. 3-5, pp. 530–540, 2007.

[8] M. C. Ece, M.Aydogdu, and V. Taskin, “Vibration of a variable cross-section beam,” Mechanics Research Communications, vol. 34, no. 1, pp. 78–84, 2007.

[9] Y. J. Shin, K. M. Kwon, and J. H. Yun, “Vibration analysis of a circular arch with variable cross section using differential transformation and generalized differential quadrature,” Journal of Sound and Vibration, vol. 309, no. 1-2, pp. 9–19, 2008.

[10] F. Romano and G. Zingone, “Deflections of beams with varying rectangular cross section,” Journal of Engineering Mechanics, vol. 118, no. 10, pp. 2128–2134, 1992.

[11] C. Franciosi and M. Mecca, “Some finite elements for the static analysis of beams with varying cross section,” Computers and Structures, vol. 69, no. 2, pp. 191–196, 1998.

[12] S. Naguleswaran, “Vibration of an Euler-Bernoulli beam on elastic end supports and with up to three step changes in cross-section,” International Journal of Mechanical Sciences, vol. 44, no. 12, pp. 2541–2555, 2002.

[13] C. Maurini,M. Porfiri, and J. Pouget, “Numerical methods for modal analysis of stepped piezoelectric beams,” Journal of Sound and Vibration, vol. 298, no. 4-5, pp. 918–933, 2006.

[14] M. Kisa andM. A. Gurel, “Free vibration analysis of uniform and stepped cracked beams with circular cross sections,” International Journal of Engineering Science, vol. 45, no. 2–8, pp. 364–380, 2007.

[15] J. Lee and L. A. Bergman, “The vibration of stepped beams and rectangular plates by an elemental dynamic flexibility method,” Journal of Sound and Vibration, vol. 171, no. 5, pp. 617–640, 1994.

[16] S. Kukla and I. Zamojska, “Frequency analysis of axially loaded stepped beams by Green’s function method,” Journal of Sound and Vibration, vol. 300, no. 3-5, pp. 1034–1041, 2007.

[17] J. W. Jaworski and E. H. Dowell, “Free vibration of a cantilevered beam with multiple steps: comparison of several theoretical methods with experiment,” Journal of Sound and Vibration, vol. 312, no. 4-5, pp. 713–725, 2008.

[18] Z. R. Lu, M. Huang, J. K. Liu, W. H. Chen, and W. Y. Liao, “Vibration analysis of multiple-stepped beams with the composite element model,” Journal of Sound and Vibration, vol. 322, no. 4-5, pp. 1070–1080, 2009.

[19] Q. Mao and S. Pietrzko, “Free vibration analysis of stepped beams by using Adomian decomposition method,” Applied Mathematics and Computation, vol. 217, no. 7, pp. 3429–3441, 2010.

[20] T. X. Zheng and T. J. Ji, “Equivalent representations of beams with periodically variable crosssections,” Engineering Structures, vol. 33, no. 3, pp. 706–719, 2011.

[21] B. V. Sankar, “An elasticity solution for functionally graded beams,” Composites Science and Technology,vol. 61, no. 5, pp. 689–696, 2001.

[22] Z. Zhong and T. Yu, “Analytical solution of a cantilever functionally graded beam,” Composites Science and Technology, vol. 67, no. 3-4, pp. 481–488, 2007.

[23] S. Kapuria, M. Bhattacharyya, and A. N. Kumar, “Bending and free vibration response of layered functionally graded beams: a theoretical model and its experimental validation,” Composite Structures,vol. 82, no. 3, pp. 390–402, 2008.

[24] Y. A. Kang and X. F. Li, “Bending of functionally graded cantilever beam with power-law nonlinearity subjected to an end force,” International Journal of Non-Linear Mechanics, vol. 44, no. 6, pp. 696–703, 2009.

[25] J. Mur´ın, M. Aminbaghai, and V. Kutiˇs, “Exact solution of the bending vibration problem of FGM beams with variation of material properties,” Engineering Structures, vol. 32, no. 6, pp. 1631–1640, 2010.

[26] A. E. Alshorbagy, M. A. Eltaher, and F. F. Mahmoud, “Free vibration characteristics of a functionally graded beam by finite element method,” Applied Mathematical Modelling, vol. 35, no. 1, pp. 412–425, 2011.

[27] Y. Huang and X. F. Li, “A new approach for free vibration of axially functionally graded beams with non-uniform cross-section,” Journal of Sound and Vibration, vol. 329, no. 11, pp. 2291–2303, 2010.

[28] Dr. Luay S. Al-Ansari , Muhannad Al-Waily and Ali M. H. Yusif Al-Hajjar; "Experimental and Numerical Study of Crack Effect on Frequency of Simply Supported Beam "; Al-Khwarizmi Engineering Journal ; received on : Julay 2011; accepted on : January 2012.

[29] S. S. Rao 'Mechanical Vibrations' Addison-Wesley Publishing Company, (1995).

[30] Won Young Yang, Wenwu Cao, Tae-Sang Chung and John Morris; "APPLIED NUMERICAL METHODS USING MATLAB"; John Wiley & Sons, Inc.; 2005.

[31] T. Stolarski , Y.Nakasone and S. Yoshimoto, " Engineering Analysis with ANSYS Software"; Elsevier Butterworth-Heinemann ; 2006.