On some abstract stochastic fractional
differential equations

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Abstract

In this paper, we shall study the mild solution for stochastic fractional integro-differential equation of the form:

$$\frac{d^\alpha u(t)}{dt^\alpha} - A(t) u(t) = \int_0^t B(t, s) u(s) \, dW(s)$$

where $0 < \alpha < \frac{1}{2}$ for $t \in J = [0, T], \ T \leq 1$, $\{A(t), t \in J\}$ is a family of linear closed operators from $E$ into $E$. $B(t, s)$ are bounded operators from $E$ into $E$. The existence and uniqueness of the mild solution of the considered equation are established, the solution is given in terms of some probability densities.

Keywords: Stochastic integro-differential equation, fractional order.

1 Introduction

Consider the stochastic fractional integro-differential equation:

$$\frac{d^\alpha u(t)}{dt^\alpha} - A(t) u(t) = \int_0^t B(t, s) u(s) \, dW(s) \quad (1)$$

with the initial condition

$$u(0) = u_0 \quad (2)$$

$u$ is unknown evaluate function, and $W(t)$ is a standard Browning motion defined over the filtered probability space $(\Omega, F, F_t, P)$. Need the conditions:
• (c₁): \( \{ A(t), t \in J \} \) is a family of linear closed operators defined on dense set \( S_1 \) in a Banach space \( E \) into \( E \).

• (c₂): The operator \([\lambda I - A(t)]^{-1}\) exist in \( G(E) \), with \( \lambda \geq 0 \), and \( \| [\lambda I - A(t)]^{-1} \| \leq \frac{C}{\lambda} \) for each \( t \in J \), where \( C \) is a positive constant independent of both \( t \) and \( \alpha \), and \( G(E) \) denote the Banach space of all linear bounded operators in \( E \) endowed with the topology defined by the operator norm.

• (c₃): For any \( t_1, t_2 \in [0, T] \) \( \| [A(t_2) - A(t_1)]A(s)^{-1}\| \leq C \| t_2 - t_1 \| \), where \( 0 < \gamma \leq 1, C > 0 \), and the constants \( c \) and \( \gamma \) are independent of \( t_1, t_2, \) and \( s \).

• (c₄): \( u_0 \in S_1 \subset E \).

• (c₅): \( B(t, s) \) are bounded operators and continuous on \( 0 \leq t \leq T \) and \( 0 \leq s \leq t \leq T \), for \( K > 0 \) is a constant \( \| B(t, s) g \| \leq K \| g \| \), \( g \in E \), where \( \| . \| \) the norm in Banach space \( E \).

2 The Mild solution

We assume that

\[
\frac{d^\alpha u(t)}{dt^\alpha} - A(t) u(t) = V(t)
\]  \( (3) \)

Hence from \([8]\) we have

\[
u(t) = u_0 + \int_0^t \psi(t - \eta, \eta) \ U(\eta) \ A(0) u_0 d\eta + \int_0^t \psi(t - \eta, \eta) \ V(\eta) \ d\eta \\
+ \int_0^t \int_0^\eta \psi(t - \eta, \eta) \ \phi(\eta, s) \ V(s) ds d\eta
\]  \( (4) \)

Where \( \psi(t, s) = \alpha \int_0^\infty \theta^{t-1} \zeta_\alpha(\theta) \exp(-\theta s A(s)) d\theta \), where \( \zeta \) is a probability density function, more details about this a probability density function can be found \([7],[10]\).

Now substitute (3), (4) in (1) we get

\[
V(t) = \int_0^t B(t, s) u_0 dW(s) + \int_0^t \int_0^s B(t, s) \psi(s - \eta, \eta) \ U(\eta) \ A(0) u_0 d\eta dW(s)
\]
\[ + \int_0^t \int_0^s B(t, s) \psi(s - \eta, \eta) \, V(\eta) \, d\eta dW(s) \]
\[ + \int_0^t \int_0^s B(t, s) \psi(s - \eta, \eta) \, \phi(\eta, \tau) \, V(\tau) \, d\tau d\eta dW(s) \quad (5) \]

**Theorem:** If we have the conditions \( C_1 \ldots C_5 \) holds then there exist unique solution of (1),(2).

**Proof.** First we solve the integral equation (5) using the method of successive approximation,
we set:

\[ V_{n+1}(t) = \int_0^t B(t, s) u_0 dW(s) + \int_0^t \int_0^s B(t, s) \psi(s - \eta, \eta) \, U(\eta) \, A(0)u_0 d\eta dW(s) \]
\[ + \int_0^t \int_0^s B(t, s) \psi(s - \eta, \eta) \, V_n(\eta) \, d\eta dW(s) \]
\[ + \int_0^t \int_0^s B(t, s) \psi(s - \eta, \eta) \, \phi(\eta, \tau) \, V_n(\tau) d\tau d\eta dW(s) \]

where \( V_0(t) \) is the zero element in \( E \).

We have from [8]

\[ \| \psi(t - \eta, \eta) \| \leq C(t - \eta)^{\alpha - 1} \]
\[ \| \phi(t, \tau) \| \leq C(t - \tau)^{\gamma - 1} \]
\[ \| U(t) \| \leq C + Ct^\gamma, \]

where \( 0 \leq \eta, \tau \leq t - \epsilon, 0 \leq t \leq T \) for any \( \epsilon > 0 \),
we get
\[ E\| V_2(t) - V_1(t) \|^2 \leq L \frac{\Gamma(\nu+1)}{\Gamma(\nu+1)} \]
where
\[ L = 2KC_M(1 + \beta(2\gamma, \nu)) , \]
\[ \nu = 2\alpha , \]
\[ \Gamma(t) \text{ is the Gamma function} , \]
\[ \beta(t, s) \text{ is the Beta function} . \]

By induction we get
\[ E\| V_{n+1}(t) - V_n(t) \|^2 \leq L^n \frac{\Gamma(\nu+1)}{\Gamma(\nu+1)^n} . \]
Thus the series $\sum_{k=0}^{n} E\|V_{k+1}(t) - V_k(t)\|^2$, uniformly converges on $[0, T]$, consequently the solution exist.

To prove uniqueness:
Let $V(t), V^*(t)$ are two solutions of (4) we have,

$$E\|V(t) - V^*(t)\|^2 \leq \frac{2Kc}{\nu-1} \int_0^t (t-\eta)^{\nu-1} E\|V(\eta) - V^*(\eta)\|^2 d\eta$$

$$+ \frac{2KC\beta(2\gamma, \nu)}{\nu-1} \int_0^t (t-\tau)^{\nu-1} E\|V(\tau) - V^*(\tau)\|^2 d\tau$$

Let $\rho(V, V^*) = \sup_{t \in [0, T]} (\exp(-\lambda t) E\|V(t) - V^*(t)\|^2)$, and $\lambda > 0$,

$$E\|V(t) - V^*(t)\|^2 \leq \frac{2KC\beta(2\gamma, \nu)}{\nu-1} \int_0^t (t-\eta)^{\nu-1} \rho(V, V^*) d\eta$$

$$+ \frac{2KC\beta(2\gamma, \nu)}{\nu-1} \int_0^t (t-\tau)^{\nu-1} \rho(V, V^*) d\tau.$$ 

$$E\|V(t) - V^*(t)\|^2 \leq M \frac{\Gamma([\nu-1])}{\Gamma(\nu+1)},$$

where $M = \sup_{t \in [0, T]} (2KC \exp(\lambda t) \rho(V, V^*)(1 + \beta(2\gamma, \nu)))$. Now at $\nu \to \infty$ then $E\|V(t) - V^*(t)\|^2 \to 0$. It must be noticed that the stochastic fractional differential equation have an application in [2,3,4,5,9].

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**References**


