Specific Internal Energy of Relativistic Rankine-Hugoniot Equations

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Abstract — The stress energy tensor and the mean velocity vector of a simple gas are expressed in terms of the Maxwell-Boltzmann distribution function. The rest density $\rho^0$, pressure, $P$, and internal energy per unit rest mass $\varepsilon$ are defined in terms of invariants formed from these tensor quantities. It is shown that $\varepsilon$ cannot be an arbitrary function of $P$ and $\rho^0$ but must satisfy a certain inequality. Thus $\varepsilon = (1/\gamma - 1)P/\rho^0$ for $\gamma > 5/3$ is impossible. It is known that if $\varepsilon$ is given by this relation and $\gamma > 2$, then sound velocity in the medium may be greater than the velocity of light in vacuum. This difficulty is removed by the inequality mentioned above.

Index Term— Specific Internal Energy, Rankine-Hugoniot Equations.

1. Introduction

Macroscopic relativistic theories of fluid dynamics characterize the fluid by giving the internal energy per unit mass measured by an observer at rest with respect to the element of the fluid as a function of the pressure, $P$, and the rest density $\rho^0$, and also by prescribing the viscosity and the heat conductivity of the fluid [1-7]. For perfect fluids, for which the latter two quantities vanish, it follows from the work of [8] and that if

$$\varepsilon = (1/\gamma - 1)P/\rho^0$$ (1)

and $\gamma$ is a constant greater than 2, then the velocity of sound in the fluid may be greater than the velocity of light in vacuum.

For [2] it is evident from the equations in [9] gives for the flow of heat in a gas at rest (Eqs. (43) and (44)) that if $\varepsilon$ is taken as given by (1) then the velocity of propagation is greater than the velocity of light in vacuum.

The purpose of this paper to show how equations of the type of (1) are ruled out on the basis of the kinetic theory of gases when this theory is formulated relativistically.

2. Basic Equations

We shall derive the equations governing the motion of the fluid considered a collection of a number of particles with rest mass $m$. Let

$$\eta^i = \frac{v^i}{\sqrt{1 - (v/c)^2}}, \quad v^i = \frac{\eta^i}{\sqrt{1 + (\eta/c)^2}}$$ (2)

Where

$$v^2 = \sum_{i=1}^{3} (v^i)^2, \quad \eta^2 = \sum_{i=1}^{3} (\eta^i)^2$$

and $v^i$ are the components of the velocity of a particle. Let $f(x,t,\eta)$ be the number of particles in the region $x^i$ to $x^i + dx^i$ in space at time $t$ and with values of $\eta^i$ between $\eta^i$ and $\eta^i + d\eta^i$. Then the Boltzman equation for $f$ is

$$Df = \frac{\partial f}{\partial t} + \frac{\eta^i}{\sqrt{1 + (\eta/c)^2}} \frac{\partial f}{\partial \eta^i} + F^i \frac{\partial f}{\partial \eta^i} = \Delta_0 f$$ (3)

Where $F^i$ is the external force per unit mass and $\Delta_0 f$ is the time rate of change in $f$ due to encounters between the particles.

An integration by parts and the integrations are carried out over the entire volume of the $\eta^1, \eta^2, \eta^3$ space.

The laws of conservation of mass, energy, and momentum follow from Equations (3) and the equation result after integration by observing that

$$\int \varphi^0 \Delta_0 f \, d\eta = 0, \quad q = 1, \ldots, 4$$

Where $\varphi^i = m, \varphi^i = m\eta^i$ (i = 1,2,3) and $\varphi^4 = m(1 + \eta^2/c^2)^{1/2}$.

Multiplying equation (3) by the various $\varphi^\theta$ in turn and integrating over all $\eta$'s we obtain five equations which may be written as

$$mU^\alpha \cdot \alpha = 0,$$ (4)

$$T^\alpha \beta \cdot \beta = 0, \quad \alpha, \beta = 1,2,3,4,$$ (5)

respectively.

$U^\alpha$ is the mass current four vector given by

$$U^\alpha = \int V^\alpha(\eta) f(\eta) \frac{d\eta}{(1 + \eta^2/c^2)^{1/2}} = \int V^\alpha(\eta) d\mu(\eta)$$ (6)

Where $V^i = \eta^i/c$, $V^4 = (1 + \eta^2/c^2)^{1/2}$; hence

$$V^\alpha V^\beta g_{\alpha \beta} = V^\alpha V_\alpha = -1.$$ (7)
Where $g_{\alpha \beta} = 0$, $\alpha \neq \beta$ and $g_{11} = g_{22} = g_{33} = -g_{44} = 1$

That is,

$$U^i = \int f(\eta) d_2 \eta \Rightarrow (1 + \eta^2 / c^2)^{1/2} d_\mu(\eta) = n,$$

$$U_i = \int \frac{\eta^j}{c} (1 + \eta^2 / c^2)^{1/2} f(\eta) d_2 \eta = \int \eta^j / c d_\mu(\eta)$$

$$= \pi \frac{\eta^j}{(1 + \eta^2 / c^2)^{1/2}}, \quad i = 1,2,3$$

$T^{\alpha \beta}$ is the stress energy tensor and it is defined as

$$T^{\alpha \beta} = mc^2 \int V^\alpha(\eta) V^\beta(\eta) \frac{f(\eta)}{(1 + \eta^2 / c^2)^{1/2}} d_2 \eta \quad (8)$$

$$\equiv mc^2 \int V^\alpha(\eta) V^\beta(\eta) d_\mu(\eta)$$

$F^\alpha$ is the four dimensional force vector:

$$F^\alpha = F^i u^i / c,$$

$$u^i = \int \frac{\eta^j}{(1 + \eta^2 / c^2)^{1/2}} f(\eta) d_2 \eta / f(\eta) d_2 \eta, \quad (9)$$

$$F^i = \frac{F^4}{(1 - u^2 / c^2)^{1/2}},$$

and $\rho^0$ is the rest density of the gas defined as

$$(\rho^0)^2 = -m^2 U^\alpha U_\alpha \quad (10)$$

It follows from equation (7) that

$$T^{\alpha \beta} = -mc^2 \int d_\mu(\eta) \quad (11)$$

It is evident from equation (5) and the fact that $\rho^0 F^\alpha$ is a four dimensional vector that $T^{\alpha \beta}$ is a tensor. From equation (8) it then follows that $f(x, \eta)$ is a scalar function under Lorentz transformations since the Lorentz invariant volume measure in $\eta$ space is

$$\frac{d_2 \eta}{(1 + \eta^2 / c^2)^{1/2}}.$$  

3. SPECIFIC INTERNAL ENERGY

We define the function $\varepsilon$ in terms of the components of $T^{\alpha \beta}$ and $U^\alpha$ as is done by [9]. We write

$$T^{\alpha \beta} = \frac{m^2 \omega}{(\rho^0)^2} U^\alpha U_\beta + \frac{m}{\rho^0} U^\alpha W^\beta$$

$$+ \frac{m}{\rho^0} U_\beta W^\alpha + W^{\alpha \beta} \quad (12)$$

Where $W^\alpha$ is the heat flow vector and $W^{\alpha \beta}$ is the stress tensor. We require that

$$W^\alpha U_\alpha = 0, \quad W^{\alpha \beta} U_\beta = 0 \quad (13)$$

The scalar $\omega$ is the energy density as measured by some one instantaneously at rest with respect to an element of the fluid.

From equation (12) we have

$$T^{\alpha \beta} = 3P - \omega \quad (14)$$

Where $W^\alpha = 3P$ and $P$ is the hydrostatic pressure. It follows from (12), (13), and (10) that

$$\omega = m^2 T^{\alpha \beta} \frac{U^\alpha U^\beta}{(\rho^0)^2} \equiv \rho^0 (c^2 + \varepsilon) \quad (15)$$

The last of these equations will be regarded as our definition of $\varepsilon$, the internal energy per unit rest mass of the fluid.

In particular the functions of the type (1) with $\gamma \geq 5 / 3$ are not permitted. Thus the restrictions on the types of functions $\varepsilon(P, \rho^0)$ have been shown to be furnished by kinetic theory.

REFERENCES