Derivation of the Nonlinear Schrödinger Equation by The Derivative Perturbation Expansion Method

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***Abstract*— The Zonal flow is an important phenomenon in the physics of plasma fusion. Due to its shearing property, it suppresses turbulence in plasma and thus enhance its confinement. This enhancement in confinement time increases the likelihood of a feasible fusion reaction. One possible candidate for the generation of the zonal flow is the modulational instability of parental drift-waves. In this work we derive the Nonlinear Schrödinger equation from the Hasegawa-Mima equation, an equation for drift-waves, using the derivative perturbation expansion method. Stability analysis on the Nonlinear Schrödinger equation yields the condition for the occurrence of modulational instability.**

***Index Term*— Zonal Flow, Drift-wave, Modulational instability, Multiple-scale Perturbation, Derivative Perturbation Expansion, Nonlinear Schrödinger Equation**

I. INTRODUCTION

IN the field of plasma physics, the zonal flow[1] refers to a mean poloidal flow with strong variation along the minor radius of the toroidal confinement. (shear property). This phenomenon is important in toroidal confinement because it suppresses turbulence and enhances confinement time[2]. As was predicted by Hasegawa, Mima, Maclennan, and Kodama[3], [4] the zonal flows can be generated from drift-wave turbulence due to the condenstion of energy at low wavenumbers. It is made possible by the conservation of enstrophy, a law particular to two-dimensional flows. The zonal flow is believed to be spontaneously generated from the drift-waves due to modulational instability of drift waves [5], [6], [7], [8],[9], [10], [11].

A generic equation that can described the modulational instability of a nonlinear wave is the Nonlinear Schrödinger equation. This equation can be obtained from Hasegawa-Mima equation[3], which models the drift waves, by performing a multiple-scale perturbation expansion. Two different variants of multiple scales method has been used in early works by Mima-Lee, Shivamoggi, and Majumdar[12], [13], [14]. The first two workers has used the reductive perturbation method [15], [16], while the last has used the derivative-expansion method [17]. They have systematically derived Nonlinear Schrödinger equation and thus showed that drift waves can undergo modulational instability leading to a spontaneous generation of zonal flows. To provide the necessary nonlinear frequency shift, a scalar nonlinearity term has been added in the Hasegawa-Mima equation. Champeaux and Diamond[1] performed the reductive perturbation method to the Hasegawa-Wakatani equation by taking into account the correct adiabatic response. In this this work the nonlinear Schrödinger equation will be derived using the derivative perturbation expansion method. This particular method allows one to look at the dynamics at different order of scale. We expect that the nonlinear Schrödinger equation comes out naturally from the Hasegawa-Mima equation by ”zooming out” into larger scale of dynamics without having to add the nonlinear shift term by hand. Notwithstanding that the Hasegawa-Mima equation is an approximate equation, in the present analysis we shall treat it as an exact equation.

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II. DERIVATIVE PERTURBATION ANALYSIS

In the framework of derivative-expansion method, the variables involved are expanded as sets of independent variables[17]: x_0, x_1; $x_2 \ldots x_n$, y_0, y_1, $y_2 \ldots y_n$, and t_0, t_1, $t_2 \ldots t_n$; where $x_n = \epsilon^n x$, $y_n = \epsilon^n y$, and $t_n = \epsilon^n t$.

The dependent variable are thus expressed as a function of those sets of independent variables:

$$\psi(x,y,t) = \psi(x_0, x_1, x_2 \ldots x_n, y_0, y_1, y_2 \ldots y_n, t_0, t_1, t_2 \ldots t_n)$$

and the derivative operators are expanded as:

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial x_0} + \epsilon \frac{\partial}{\partial x_1} + \epsilon^2 \frac{\partial}{\partial x_2} + \cdots + \epsilon^n \frac{\partial}{\partial x_n},$$

$$\frac{\partial}{\partial y} = \frac{\partial}{\partial y_0} + \epsilon \frac{\partial}{\partial y_1} + \epsilon^2 \frac{\partial}{\partial y_2} + \cdots + \epsilon^n \frac{\partial}{\partial y_n},$$

and

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial t_0} + \epsilon \frac{\partial}{\partial t_1} + \epsilon^2 \frac{\partial}{\partial t_2} + \cdots + \epsilon^n \frac{\partial}{\partial t_n}.$$

A further assumption is made that the perturbation can be represented as a series of perturbations with different scales. This representation is expressed as a power series of a small parameter ϵ:

$$\psi(x_0,\dots x_N,y_0,\dots y_N,t_0,\dots t_N) = \sum_{n=0}^{N}\epsilon^n\psi_n(x_0,\dots x_N,y_0,\dots y_N,t_0,\dots t_N).$$

We now perform the analysis on the modified Hasegawa-Mima equation

$$(\partial_t + v_*\partial_y)\tilde{\psi} + \partial_x\overline{\psi}\partial_y\tilde{\psi} - (\partial_t + \partial_x\psi\partial_y - \partial_y\psi\partial_x)[\rho_s^2(\partial_x^2+\partial_y^2)\psi] = 0, \quad (1)$$

where $\overline{\psi}$ is the y-averaged component of ψ and $\tilde{\psi}$ the remaining part:

$$\tilde{\psi} = \psi - \overline{\psi}.$$

We expand $\tilde{\psi}$ and $\overline{\psi}$ into the following function

$$\tilde{\psi} = \sum_{n=0}^{N}\varepsilon^n\tilde{\psi}_n(x_0\dots x_N, y_0\dots y_N, t_0\dots t_N);$$
$$\overline{\psi} = \sum_{n=0}^{N}\varepsilon^n\overline{\psi}_n(x_0\dots x_N, y_0\dots y_N, t_0\dots t_N). \quad (2)$$

Being a y-averaged function, $\overline{\psi}$ is independent of y. $\overline{\psi}$ is also assumed to be very slowly varying as compared to $\tilde{\psi}$. By substituting the expansion into the Modified Hasegawa-Mima Equation (1), the equation gives a hierarchy of equations at different orders of ϵ. The Modified Hasegawa-Mima equation must be satisfied at every order of ϵ. We examine the equation at subsequent orders.

<u>Order (ϵ)</u>

At this order the equation assumes the following form

$$[1-\rho_s^2(\partial_{x_0}^2+\partial_{y_0}^2)]\partial_{t_0}^2\tilde{\psi}_1 + v_*\partial_{y_0}\tilde{\psi}_1 = 0. \quad (3)$$

This equation admits solutions of the form:

$$\tilde{\psi}_1 = A_1(x_1,x_2,\dots,x_n,y_1,y_2,\dots,y_n,t_1,t_2,\dots,t_n) \times \exp[i(k_x x_0 + k_y y_0 - \omega t_0)] + c.c., \quad (4)$$

This solution gives the dispersion relation for drift waves

$$\omega = \frac{v_* k_y}{1+\rho_s^2}. \quad (5)$$

upon substitution into (3). The drift waves will propagate at the group velocity $\mathbf{v}_g \equiv {\partial\omega}/{\partial\mathbf{k}}$

$$\frac{\partial\omega}{\partial\mathbf{k}} = \frac{v_*\hat{y}}{1+\rho_s^2k^2} - \frac{2\rho_s^2\omega\mathbf{k}}{1+\rho_s^2k^2}. \quad (6)$$

Using equation (4) to substitute for $\tilde{\psi}_1$, we go to the second order in ϵ.

<u>Order (ϵ^2)</u>

$$e^{i\theta}\left(\frac{\partial A_1}{\partial t_1} + \rho_s^2k^2\frac{\partial A_1}{\partial t_1} - 2\omega k_y\rho_s^2\frac{\partial A_1}{\partial y_1} + v_*\frac{\partial A_1}{\partial y_1} - 2\omega k_x\rho_s^2\frac{\partial A_1}{\partial x_1}\right) + \left[1-\rho_s^2\left(\frac{\partial^2}{\partial^2_{x_0}}+\frac{\partial^2}{\partial^2_{y_0}}\right)\right]\frac{\partial\tilde{\psi}_2}{\partial t_0} + v_*\frac{\partial\tilde{\psi}_2}{\partial y_0} = 0, \quad (7)$$

where $\theta \equiv k_x x_0 + k_y y_0 - \omega t_0$ and $k^2 = k_x^2 + k_y^2$.

The terms with the factor of $e^{i\theta}$ will give a secular solution to the equation. They shall be removed by setting them to zero:

$$\frac{\partial A_1}{\partial t_1} + \rho_s^2k^2\frac{\partial A_1}{\partial t_1} - 2\omega k_y\rho_s^2\frac{\partial A_1}{\partial y_1} + v_*\frac{\partial A_1}{\partial y_1} - 2\omega k_x\rho_s^2\frac{\partial A_1}{\partial x_1} = 0. \quad (8)$$

Taking Eq. (6) into account we can simplify Eq. (8) into the following relation

$$\left(1+\rho_s^2k^2\right)\left\{\frac{\partial A_1}{\partial t_1} + \left(\frac{\partial\omega}{\partial\mathbf{k}}\bullet\nabla_1\right)A_1\right\} = 0. \quad (9)$$

This equation shows that the local change of the amplitude of the drift wave is due to propagation of the envelope with the group velocity of the drift wave.

Having removed the secular terms, we are left with:

$$\left[1-\rho_s^2\left(\frac{\partial^2}{\partial^2_{x_0}}+\frac{\partial^2}{\partial^2_{y_0}}\right)\right]\frac{\partial\tilde{\psi}_2}{\partial t_0} + v_*\frac{\partial\tilde{\psi}_2}{\partial y_0} = 0. \quad (10)$$

This equation suggests an operator exactly identical to that which acts on $\tilde{\psi}_1$ (3). We can assume that the solution to (10) is absorbed in $\tilde{\psi}_1$, hence $\tilde{\psi}_2$ can be set to 0.

III. Nonlinear Schrödinger Equation

Proceeding to the next order in ϵ, we obtain

<u>Order (ϵ^3)</u>:

$$e^{i\theta}\left[ik_yA_1\frac{\partial\overline{\psi}_1}{\partial x_1}+ik_x^2k_y\rho_s^2A_1\frac{\partial\overline{\psi}_1}{\partial x_1}+ik_y^3\rho_s^2A_1\frac{\partial\overline{\psi}_1}{\partial x_1}+\frac{\partial A_1}{\partial t_2}+k_x^2\rho_s^2\frac{\partial A_1}{\partial t_2}+k_y^2\rho_s^2\frac{\partial A_1}{\partial t_2}-2\omega k_y\rho_s^2\frac{\partial A_1}{\partial y_2}\right]$$
$$+e^{i\theta}\left[v_*\frac{\partial A_1}{\partial y_2}-2ik_y\rho_s^2\frac{\partial^2A_1}{\partial y_1\partial t_1}+i\omega\rho_s^2\frac{\partial^2A_1}{\partial y_1^2}-2\omega k_x\rho_s^2\frac{\partial A_1}{\partial x_2}-2ik_x\rho_s^2\frac{\partial^2A_1}{\partial x_1\partial t_1}+i\omega\rho_s^2\frac{\partial^2A_1}{\partial x_1^2}\right]$$
$$+\left[1-\rho_s^2\left(\frac{\partial^2}{\partial_{x_0}^2}+\frac{\partial^2}{\partial_{y_0}^2}\right)\right]\frac{\partial\tilde{\psi}_3}{\partial t_0}+v_*\frac{\partial\tilde{\psi}_3}{\partial y_0}=0, \qquad (11)$$

where we have assumed $\tilde{\psi}_2=0$ to arrive at (11).

Non-secularity condition requires that

$$ik_yA_1\frac{\partial\overline{\psi}_1}{\partial x_1}+ik_x^2k_y\rho_s^2A_1\frac{\partial\overline{\psi}_1}{\partial x_1}+ik_y^3\rho_s^2A_1\frac{\partial\overline{\psi}_1}{\partial x_1}+\frac{\partial A_1}{\partial t_2}+k_x^2\rho_s^2\frac{\partial A_1}{\partial t_2}+k_y^2\rho_s^2\frac{\partial A_1}{\partial t_2}-2\omega k_y\rho_s^2\frac{\partial A_1}{\partial y_2}+v_*\frac{\partial A_1}{\partial y_2}-2ik_y\rho_s^2\frac{\partial^2A_1}{\partial y_1\partial t_1}+i\omega\rho_s^2\frac{\partial^2A_1}{\partial y_1^2}-2\omega k_x\rho_s^2\frac{\partial A_1}{\partial x_2}-2ik_x\rho_s^2\frac{\partial^2A_1}{\partial x_1\partial t_1}+i\omega\rho_s^2\frac{\partial^2A_1}{\partial x_1^2}=0. \qquad (12)$$

The equation above can be simplified into

$$\left[\frac{\partial A_1}{\partial t_2}+(\mathbf{v}_g\bullet\nabla_2)A_1\right]+i\frac{\rho_s^2}{1+\rho_s^2k^2}\left[(\omega+2k_xv_{g_x})\frac{\partial^2A_1}{\partial x_1^2}+(\omega+2k_yv_{g_y})\frac{\partial^2A_1}{\partial y_1^2}+2(k_xv_{g_y}+k_yv_{g_x})\frac{\partial^2A_1}{\partial x_1\partial y_1}\right] \qquad (13)$$
$$ik_yA_1\frac{\partial\overline{\psi}_1}{\partial x_1}=0.$$

To obtain (13) we have used (9) to substitute for $\frac{\partial A_1}{\partial t_1}$. The equation shows that the last term is nonlinear, and is thus accounted for the nonlinear frequency shift. The factor $\frac{\partial\overline{\psi}_1}{\partial t_1}$ is to be determined from the equation at the next higher order.

When the secular terms has been removed, we obtain an equation for $\overline{\psi}_3$ which involved first order equations:

$$\left[1-\rho_s^2\left(\frac{\partial^2}{\partial_{x_0}^2}+\frac{\partial^2}{\partial_{y_0}^2}\right)\right]\frac{\partial\tilde{\psi}_3}{\partial t_0}+v_*\frac{\partial\tilde{\psi}_3}{\partial y_0}=0 \qquad (14)$$

Again, as the case with $\tilde{\psi}_2$, we set $\tilde{\psi}_3=0$. With this value, the equation at the order ϵ^4 is

Order (ϵ^4):

$$e^{i\theta}\left[ik_yA_1\frac{\partial\overline{\psi}_1}{\partial x_2}+ik_x^2k_y\rho_s^2A_1\frac{\partial\overline{\psi}_1}{\partial x_2}+ik_y^3\rho_s^2A_1\frac{\partial\overline{\psi}_1}{\partial x_2}+ik_yA_1\frac{\partial\overline{\psi}_2}{\partial x_1}+ik_x^2k_y\rho_s^2A_1\frac{\partial\overline{\psi}_2}{\partial x_1}+ik_y^3\rho_s^2A_1\frac{\partial\overline{\psi}_2}{\partial x_1}\right]$$
$$+e^{i\theta}\left[\frac{\partial\overline{\psi}_1}{\partial x_1}\frac{\partial A_1}{y_1}+k_x^2\rho_s^2\frac{\partial\overline{\psi}_1}{\partial x_1}\frac{\partial A_1}{\partial y_1}+3k_y^2\rho_s^2\frac{\partial\overline{\psi}_1}{\partial x_1}\frac{\partial A_1}{\partial y_1}-i2k_y\rho_s^2\frac{\partial^2A_1}{\partial y_2\partial t_1}-i2k_y\rho_s^2\frac{\partial^2A_1}{\partial y_1\partial t_2}+i2\omega\rho_s^2\frac{\partial^2A_1}{\partial y_1\partial y_2}\right]$$
$$+e^{i\theta}\left[-\rho_s^2\frac{\partial^3A_1}{\partial y_1^2\partial t_1}-i2k_x\rho_s^2\frac{\partial^2A_1}{\partial x_2\partial t_1}+2k_xk_y\rho_s^2\frac{\partial\overline{\psi}_1}{\partial x_1}\frac{\partial A_1}{\partial x_1}-i2k_x\rho_s^2\frac{\partial^2A_1}{\partial x_1\partial t_2}+i2\omega\rho_s^2\frac{\partial^2A_1}{\partial x_1\partial x_2}-\rho_s^2\frac{\partial^3A_1}{\partial x_1^2\partial t_1}\right]$$
$$-\rho_s^2\frac{\partial^3\overline{\psi}_1}{\partial x_1^2\partial t_1}-4k_xk_y\rho_s^2\frac{\partial A_1}{\partial y_1}\frac{\partial A_1^*}{\partial y_1}-2k_xk_y\rho_s^2A_1^*\frac{\partial^2A_1}{\partial y_1^2}$$
$$-2k_xk_y\rho_s^2A_1\frac{\partial^2A_1^*}{\partial y_1^2}-2k_x^2\rho_s^2\frac{\partial A_1^*}{\partial y_1}\frac{\partial A_1}{\partial x_1}+2k_y^2\rho_s^2\frac{\partial A_1^*}{\partial y_1}\frac{\partial A_1}{\partial x_1}$$
$$-2k_x^2\rho_s^2\frac{\partial A_1}{\partial y_1}\frac{\partial A_1^*}{\partial x_1}+2k_y^2\rho_s^2\frac{\partial A_1}{\partial y_1}\frac{\partial A_1^*}{\partial x_1}+4k_xk_y\rho_s^2\frac{\partial A_1}{\partial x_1}\frac{\partial A_1^*}{\partial x_1}$$
$$-2k_x^2\rho_s^2A_1^*\frac{\partial^2A_1}{\partial x_1\partial y_1}+2k_y^2\rho_s^2A_1^*\frac{\partial^2A_1}{\partial x_1\partial y_1}-2k_x^2\rho_s^2A_1\frac{\partial^2A_1^*}{\partial x_1\partial y_1}$$
$$+2k_y^2\rho_s^2A_1\frac{\partial^2A_1^*}{\partial x_1\partial y_1}+2k_xk_y\rho_s^2A_1^*\frac{\partial^2A_1}{\partial x_1^2}+2k_xk_y\rho_s^2A_1\frac{\partial^2A_1^*}{\partial y_1^2}=0 \qquad (15)$$

Having removed the secular terms, we obtain

$$\frac{\partial^2}{\partial x_1^2}\left(\frac{\partial\overline{\psi}_1}{\partial t_1}\right)=\left[2k_xk_y\left(\frac{\partial^2}{\partial x_1^2}-\frac{\partial^2}{\partial y_1^2}\right)+2\left(k_x^2-k_y^2\right)\frac{\partial^2}{\partial x_1\partial y_1}\right]|A_1|^2. \qquad (16)$$

The equation shows relation of zonal component and the amplitude of the fluctuating part. Considering that $\overline{\psi}_1$ is a surface averaged function, it is independent of y. Consequently, the above equality always holds only when the r.h.s is also independent of y. This can be satisfied by averaging the r.h.s over the surface (y coordinate).

Thus, we have

$$\frac{\partial^2}{\partial x_1^2}\left(\frac{\partial\overline{\psi}_1}{\partial t_1}\right)=2k_xk_y\frac{\partial^2}{\partial x_1^2}\left\langle|A_1|^2\right\rangle. \qquad (17)$$

Integrating (17) twice, we obtain

$$\frac{\partial \overline{\psi}_1}{\partial t_1} = 2k_x k_y \left\langle |A_1|^2 \right\rangle. \quad (18)$$

Since $\overline{\psi}_1$ is driven by A_1, we can assume that to the lowest order, $\overline{\psi}_1$ is a function of $x_1 - v_{g_x} t_1$. Hence we can make a substitution of $\frac{\partial \overline{\psi}_1}{\partial x_1}$ for $\frac{\partial \overline{\psi}_1}{\partial t_1}$ in (18). Thus we have

$$\frac{\partial \overline{\psi}_1}{\partial x_1} = -\frac{1}{v_{g_x}} 2k_x k_y \left\langle |A_1|^2 \right\rangle. \quad (19)$$

By inserting (19) into (13) and taking into account that the nonlinear frequency shift affects only the dynamics in the *x*-direction, the NLS equation reduces to one dimensional Nonlinear Schrödinger equation:

$$i\left[\frac{\partial A_1}{\partial t_2} + (\mathbf{v}_g \bullet \nabla_2) A_1\right] - \frac{\rho_s^2}{1+\rho_s^2 k^2}(\omega + 2k_x v_{g_x})\frac{\partial^2 A_1}{\partial x_1^2} + \frac{1}{v_{g_x}} 2k_x k_y^2 \left\langle |A_1|^2 \right\rangle A_1 = 0. \quad (20)$$

An amplitude modulated wave can be represented as the sum of an unmodulated carrier wave and the upper and lower sidebands.

$$A_1 = A_0 \exp(-i\Delta\omega_0 t) \times \left\{1 + a_+ \exp(i\mathbf{K} \bullet \mathbf{r} - i\Omega t) + a_-^* \exp(-i\mathbf{K} \bullet \mathbf{r} - i\Omega^* t)\right\}, \quad (21)$$

where $\Delta\omega_0 \equiv \Delta\omega[|A_0|]$. The modulation is unstable when the coefficient of the dispersion and the nonlinear term have the same sign[18],[19]:

$$\left\{-\frac{\rho_s^2}{1+\rho_s^2 k^2}(\omega + 2k_x v_{g_x})\right\}\frac{1}{v_{g_x}} 2k_x k_y^2 \left\langle |A_1|^2 \right\rangle > 0, \quad (22)$$

which is satisfied when

$$1 - 3\rho_s^2 k_x^2 + \rho_s^2 k_y^2 > 0. \quad (23)$$

The criterion agrees well with that found by Smolyakov et al.[7] for modulation waves with zonal phase fronts.

If the criterion (23) is fulfilled, the growth rate curve, Γ^2 vs K^2, is an inverted parabola with maximum at

$$\Gamma_{\max} = \Delta\omega_0, \qquad K_{\max} = \frac{(1+\rho_s^2 k^2)^{5/2}}{(1 - 3\rho_s^2 k_x^2 + \rho_s^2 k_y^2)^{1/2}} \frac{|A|}{\rho_s^2 v_*}. \quad (24)$$

Thus it has been shown that these results are equivalent to those obtained using the Reynolds averaging method[5]. It confirms that the multiple scale perturbation analysis is indeed an alternative method to the averaging method[20].The comparison of the Reynolds averaging method and the multiple scale perturbation analysis has been studied in more details in recent paper by Smith[21] for cases of modulations in incompressible fluids.

IV. Conclusion

We have derived a nonlinear Schrödinger equation for modulations on a train of drift or Rossby waves in a very universal, if heuristic, fashion. The nonlinear Schr¨odinger equation has been widely studied in other applications and is known to have soliton solutions. However, we have analyzed it only for stability to small modulations and have found criteria in agreement with those found by Smolyakov et al.[7] for modulation waves with zonal phase fronts. Our results are encouraging as a step towards explaining the experimental discovery by Shats and Solomon[22] of modulational instability associated with low-frequency zonal flows, but the Hasegawa–Mima equation is rather too simplified for direct comparison with experiment.

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