A Comparison Among Homotopy Perturbation Method And The Decomposition Method With The Variational Iteration Method For Dispersive Equation

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Abstract--- In this article, we implement a relatively new numerical technique and we present a comparative study among Homotopy perturbation method and Adomian decomposition method, the variational iterative method. These methods in applied mathematics can be an effective procedure to obtain for approximate solutions. The study outlines the significant features of the three methods. The analysis will be illustrated by investigating the homogeneous Dispersive equation model problem. This paper is particularly concerned a numerical comparison with the Adomian decomposition and Homotopy perturbation method, the variational iterative method The numerical results demonstrate that the new methods are quite accurate and readily implemented.

Index Term--- Dispersive Equation, The Decomposition Method, Homotopy perturbation Method, The variational iterative Method

I. INTRODUCTION

Partial differential equations which arise in real-world physical problems are often too complicated to be solved exactly. And even if an exact solution is obtainable, the required calculations may be too complicated to be practical, or it might be difficult to interpret the outcome. Very recently, some promising approximate analytical solutions are proposed, such as Exp-function method [1–2], Adomian decomposition method [3–7], variational iteration method [8–10] and Homotopy-perturbation method [11–16].

Other methods are reviewed in Refs.[17–18].

HPM is the most effective and convenient on ef or both linear and nonlinear equations. This method does not depend on a small parameter. Using homotopy technique in topology, a homotopy is constructed with an embedding parameter \( p \in [0, 1] \), which is considered as a “small parameter”. HPM has been shown to effectively, easily and accurately solve a large class of linear and nonlinear problems with components converging rapidly to accurate solutions. HPM was first proposed by He [11] and was successfully applied to various engineering problems [19–21].

Recently, VIM is applied for exact solutions of Dispersive Equation [22–27]. The aim of this work is to employ HPM and ADM to obtain the exact solutions for Dispersive Equations and to compare the results with those of VIM. Different from ADM, where specific algorithms are usually used to determine the Adomian polynomials, HPM handles linear and nonlinear problems in a simple manner by deforming a difficult problem into a simple one.

Dispersion equation has the following form:

\[
 u_t + u_{xx} = 0, \quad x \in (-\infty, \infty), \quad t \geq 0,
\]

(1)

with the boundary and initial conditions

\[
 u(x, 0) = \cos(\pi x)
\]

(2)

where \( u(x, t) \neq c \text{ars} k \pi^3 \) are given functions. These equations appear in such diverse phenomena as: elastic waves in solid including vibrating string, bars, membranes, sound or acoustics, electromagnetic waves and nuclear reactors.

II. NUMERICAL METHODS

1. Fundamentals of the Homotopy Perturbation Method

To illustrate the basic ideas of this method, we consider the following equation [11]:

\[
 A(u) - f(r) = 0, \quad r \in \Omega,
\]

(3)

with boundary condition

\[
 B(\frac{\partial u}{\partial n}) = 0, \quad r \in \Gamma,
\]

(4)

where \( A \) is a general differential operator, \( B \) a boundary operator, \( f \) a known analytical function and \( \Gamma \) is the boundary of the domain \( \Omega \).

\( A \) can be divided into two parts which are \( L \) and \( N \), where \( L \) is linear and \( N \) is nonlinear. Eq. (3) can therefore be rewritten as follows:

\[
 L(u) + N(u) - f(r) = 0, \quad r \in \Omega,
\]

(5)

Homotopy perturbation structure is shown as follows:

\[
 H(\nu; \nu_0) = \left(1 - p \right) [ L(\nu) - L(\nu_0)] + p [ A(\nu) - f(r)] = 0,
\]

(6)

where

\[
 \nu(r, p): \Omega \times [0,1] \rightarrow \Re.
\]

(7)

In Eq. (6), \( p \in [0, 1] \) is an embedding parameter and \( \nu_0 \) is the first approximation that satisfies the boundary condition.
We can assume that the solution of Eq. (6) can be written as a power series in \( p \), as following:

\[
v = v_0 + pv_1 + p^2v_2 + p^3v_3 + \cdots,
\]

and the best approximation for solution is

\[
u = \lim_{p \to 1} v = v_0 + v_1 + v_2 + v_3 + \cdots,
\]

The above convergence is discussed in [11].

2. Adomian Decomposition Method

The decomposition series method does not require this discrimination and resulting massive computation. In this work we apply the second method to obtain analytic and approximate solutions of the equation from (1), and using the decomposition method, the equation (1) is approximated by the operators in the following form

\[
\begin{align*}
    u_t + u_{xxx} &= 0, \\
    u_t &= -u_{xxx}, \\
    u_t &= f(x,t), \\
    L^{-1}_t(u) &= L^{-1}_t[f(x,t)], \\
    u(x,t) &= -L^{-1}_t[L_s u],
\end{align*}
\]

(10)

where \( L_s = \frac{\partial^3}{\partial x^3} \) respectively. Assuming the inverse of the operator \( L^{-1}_t = \int_0^t \) exists, and it can conveniently be integrated with respect to \( x \) from 0 to \( x \), then applying the inverse operator \( L^{-1}_t \) to (10) yields

\[
u(x,t) = -L^{-1}_t[f(x,t)],
\]

(11)

Therefore, it follows that

\[
u(x,t) = u(0,t) - L^{-1}_t(L_s u),
\]

(12)

The zeroth components is obtained, by using the initial condition, as

\[
u = u_0(x,0) + u_{n+1},
\]

(13)

\[
u_{n+1} = -L^{-1}_t(L_s u_n), \\
u_0 = u(x,0) = \cos(\pi x), \\
u_0 = \cos(\pi x),
\]

(14)

The remaining components \( u_n(x,t), n \geq 1 \) can be completely determined such that each term is computed by using the previous term. Since \( u_0 \) is know,

\[
u_1 = -L^{-1}_t(L_s u_0) = \int_0^t [L_s u_0] dt, \\
u_2 = -L^{-1}_t(L_s u_1) = \int_0^t [L_s u_1] dt, \\
u_3 = -L^{-1}_t(L_s u_2) = \int_0^t [L_s u_2] dt, \\
u_4 = -L^{-1}_t(L_s u_3) = \int_0^t [L_s u_3] dt,
\]

(15)

As a results, the series solutions is given by

\[
u(x,t) = \sum_{n=0}^{\infty} u_n(x,t),
\]

(16)

where \( L^{-1}_s \) is the previously given integration operator. The solution \( u(x,y) \) must satisfy the requirements imposed by the initial conditions.

The decomposition method provides a reliable technique that requires less work if compared with the traditional techniques. Moreover, the proposed method does not need discrimination of the problem to obtain numerical results. We can evaluate the approximate solution \( \phi_y \), by using the \( \gamma \) – term approximation. That is,

\[
\phi_y = \sum_{k=0}^{\gamma-1} u_k(x,y)
\]

(17)

Where the components are produce as

\[
\begin{align*}
    \phi_1 &= u_0(x,y), \\
    \phi_2 &= u_0(x,y) + u_1(x,y), \\
    \phi_3 &= u_0(x,y) + u_1(x,y) + u_2(x,y), \\
    \phi_{\gamma} &= u_0(x,y) + u_1(x,y) + u_2(x,y) + \cdots + u_{\gamma-1}(x,y)
\end{align*}
\]
II. THE VARIATIONAL ITERATIONAL METHODS

Consider the differential equation

\[ L_1 u + Nu = g(t). \]

(18)

Where \( L_1 \) is a linear operator, \( N \) is a non-linear operator and \( g(t) \) is a known and Nonlinear analytical function. Ji Huan has modified the above method into an iteration method \([2,7]\) in the following way:

\[
\begin{align*}
 u_{n+1} &= u_n + \int_0^1 \lambda (L u_n(x) + N u_n(x) - g(x)) dx,
\end{align*}
\]

(19)

where \( \lambda \) is a general Lagrange’s multiplier, which can be identified optimally via the variational theory, and \( \tilde{u}_n \) is a restricted variation which means \( \delta \tilde{u} = 0 \).

It is obvious now that the main steps of He’s variational iteration method require first the determination of the Lagrangian multiplier \( \lambda \) that will be identified optimally. Having determined the Lagrangian multiplier, the successive approximations \( u_{n+1}, n \geq 0 \), of the solution \( u \) will be readily obtained upon using any selective function \( u_0 \). Consequently, the solution

\[ u = \lim_{n \to \infty} u_n, \text{ for } (n \to \infty). \]

(20)

In other words, correction functional \( (19) \) will give several approximations, and therefore the exact solution is obtained at the limit of the resulting successive approximations.

To give a clear overview of the methodology, we consider several examples in the following section.

Example 1:

For comparison purposes, we consider a Dispersive equation model problem in order to illustrate the technique discussed above. This problem is as follows:

\[
\begin{align*}
u_t + u_{xxx} &= 0, \quad x \in (-\infty, \infty), \quad t \geq 0, \\
u_0(x,0) &= \cos(\pi x)
\end{align*}
\]

(21)

with in initial conditions

\[ Y = \frac{\partial Y}{\partial t}, \quad Y''' = \frac{\partial^3 Y}{\partial x^3} \quad \text{and} \quad p \in [0,1]. \]

With initial approximation \( Y_0 = u_0 = \cos(\pi x) \), suppose the solution of Eq. (23) has the form:

\[
Y = Y_0 + pY_1 + p^2Y_2 + p^3Y_3 + p^4Y_4 + \cdots
\]

(24)

\[
= \sum_{n=0}^{\infty} p^n Y_n(x,t)
\]

Then, substituting Eq. (24) into Eq. (23), and rearranging based on powers of \( p \)-terms, we obtain:

\[
p^0 : Y_0 - u_0 = 0,
\]

(25)

\[
p^1 : Y_1 + Y_0 + Y_0' = 0,
\]

(26)

\[
p^2 : Y_2 + Y_1 + Y_1' = 0,
\]

(27)

\[
p^3 : Y_3 + Y_2 + Y_2' = 0,
\]

(28)

\[
p^4 : Y_4 + Y_3 + Y_3' = 0,
\]

(29)

\[
\vdots
\]

(30)

with solving Eqs. (25)-(29)

\[ Y_0 = u_0 = \cos(\pi x), \]

(31)

\[ Y_1' = -\pi^3 t \sin(\pi x), \]

(32)

\[ Y_2 = -\frac{1}{2} \pi^6 t^2 \cos(\pi x), \]

(33)

\[ Y_3 = -\frac{1}{6} \pi^9 t^3 \sin(\pi x), \]

(34)

\[ Y_4 = \frac{1}{24} \pi^{12} t^4 \cos(\pi x), \]

(35)
the above terms of the series (23) could calculated. When we consider the series (23) with the terms (30)-(34) and suppose \( p = 1 \), we obtain approximation solution of Eq. (22) as following:

\[
 u(x,t) = Y_0 + Y_1 + Y_2 + Y_3 + Y_4 + \cdots
\]

(35)

\[
 u(x,t) = \cos(\pi x) - \pi^3 t \sin(\pi x) - \frac{1}{2} \pi^6 t^2 \cos(\pi x) + \frac{1}{6} \pi^9 t^3 \sin(\pi x) + \frac{1}{24} \pi^{12} t^4 \cos(\pi x) + \cdots
\]

As a result, the components \( Y_0, Y_1, Y_2, \cdots \) are identified and the series solution thus entirely determined. In order to solve this equation by using the Adomian decomposition method, we simply take the equation in an operator form.

Following the same manner as given by equation (13) to find the zeroth component of \( u_0 \) as

\[
 u(x,0) = u_0 = \cos(\pi x),
\]

(36)

and remaining components \( u_1, u_2, u_3 \), etc. We are computed by a recursive scheme directly by hand using (15). Some of the symbolically computed components are as follows

\[
 u_1 = -L_a^{-1}(L_u u_0) = \int_0^t \left[ L_u \left( \cos(\pi x) \right) \right] dt = -\pi^3 t \sin[\pi x],
\]

(37)

\[
 u_2 = -L_a^{-1}(L_u u_1) = \int_0^t \left[ L_u \left( -\pi^3 t \sin(\pi x) \right) \right] dt = -\frac{1}{2} \pi^6 t^2 \cos(\pi x),
\]

\[
 u_3 = -L_a^{-1}(L_u u_2) = \int_0^t \left[ L_u \left( \frac{1}{2} \pi^6 t^2 \cos(\pi x) \right) \right] dt = \frac{1}{6} \pi^9 t^3 \sin(\pi x),
\]

\[
 u_4 = -L_a^{-1}(L_u u_3) = \int_0^t \left[ L_u \left( \frac{1}{6} \pi^9 t^3 \sin(\pi x) \right) \right] dt = \frac{1}{24} \pi^{12} t^4 \cos(\pi x),
\]

\[
 u_5 = -L_a^{-1}(L_u u_4),
\]

In this manner, four component of the decomposition series were obtained of which \( u(x,y) \) was evaluated to have the following expansion.

\[
 u(x,t) = u_0 + u_1 + u_2 + u_3 + u_4 + \cdots
\]

\[
 u(x,t) = \cos(\pi x) - \pi^3 t \sin(\pi x) - \frac{1}{2} \pi^6 t^2 \cos(\pi x) + \frac{1}{6} \pi^9 t^3 \sin(\pi x) + \frac{1}{24} \pi^{12} t^4 \cos(\pi x) + \cdots
\]

As a simply,

\[
 u(x,t) = \frac{1}{24} \left( 24 - 12 \pi^6 t^2 + \pi^{12} t^4 \right) \cos(\pi x) + 4 \pi^4 t \left( -6 + \pi^6 t^2 \right) \sin(\pi x)
\]

Continuing the expansion to the last term, it may be proved that the section of the decomposition series (14) as

\[
 u(x,t) = \cos(\pi x + \pi^4 t)
\]

by using the variational iteration method, we simply take the equation in an operator form.

\[
 L_u u + Nu = g(t).
\]

and

\[
 u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda_1(u_{xxx}) \, d\xi,
\]

(38)

\[
 u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda_2(u_{xxx}) \, d\xi,
\]

(39)

where \( \lambda_i (i = 1,2) \) are general Lagrange multipliers [1] and can be identified optimally via the variational theory [18-15], \( u_0 \) is an initial approximation or trial function with possible unknowns, \( \tilde{u}_n \) are considered as restricted variations [2,7-10], i.e., \( \tilde{u} = 0 \).

Making the correction functional (39) stationary, noticing that \( \delta \tilde{u} = 0 \),

\[
 \delta u_{n+1}(x,t) = \delta u_n(x,t) + \int_0^t \lambda_1 \left( \frac{\partial^2 u_n(\xi,t)}{\partial \xi^2} + \frac{\partial^2 \tilde{u}_n(\xi,t)}{\partial \xi^2} - \tilde{u}_n(\xi,t) \right) d\xi,
\]

(40)

\[
 \delta u_{n+1}(x,t) = \delta u_n(x,t) + \int_0^t \lambda_2 \left( \frac{\partial^2 u_n(\xi,t)}{\partial \xi^2} \right) d\xi,
\]

(41)

\[
 \delta u_{n+1}(x,t) = \delta u_n(x,t)(1 - \lambda_1(t)) \delta \xi + \int_0^t \lambda_2(\xi,t) \delta \xi - \lambda_2(\xi,t) \delta \xi = 0,
\]

yields the following stationary conditions.

\[
 \delta u_n : 1 - \lambda_2(\xi) = 0,
\]

\[
 \delta \frac{\partial u_n}{\partial \xi} : \lambda_2(\xi) = 0,
\]

\[
 \delta u_n : \lambda_2(\xi) = 0.
\]

The Lagrange multiplier, therefore can be readily identified, \( \lambda_2(\xi) = \xi - x \) and by the same manipulation, we have \( \lambda_1(\tau) = \tau - t \).
As a result, we obtain the following iteration formulae in y - and x- directions:

\[ u_{n+1}(x, y) = u_n(x, y) + \int_0^\gamma (\tau - y) \left( \frac{\partial^2 u_n(x, \tau)}{\partial \tau^2} + \frac{\partial^2 u_n(x, \tau)}{\partial \tau^2} - u_n(x, \tau) \right) d\tau, \]

\[ u_{n+1}(y, x) = u_n(y, x) + \int_0^\gamma (\xi - x) \left( \frac{\partial^2 u_n(y, \xi)}{\partial \xi^2} + \frac{\partial^2 u_n(y, \xi)}{\partial \xi^2} - u_n(y, \xi) \right) d\xi, \]

Now we begin with an arbitrary initial approximation: \( u_0 = A + Bx \), where A and B are constants in x to be determined by using the initial conditions (22), thus we have

\[ u_0 = \cos\left(\pi x\right) \]

By the above variational iteration formula in x-direction (42), we can obtain following result:

\[ u_1(x, y) = u_0(x, y) + \int_0^\gamma (\xi - x) \left( \frac{\partial^2 u_0(x, \xi)}{\partial \xi^2} + \frac{\partial^2 u_0(x, \xi)}{\partial \xi^2} - u_0(x, \xi) \right) d\xi, \]

\[ u_1(y, x) = u_0(y, x) + \int_0^\gamma (\xi - x) \left( \frac{\partial^2 u_0(y, \xi)}{\partial \xi^2} + \frac{\partial^2 u_0(y, \xi)}{\partial \xi^2} - u_0(y, \xi) \right) d\xi, \]

and so on, in the same manner the rest of components of the iteration formula (42) were obtained using the Mathematica Package.

The Solution \( u(x, y) \) in a closed form is

\[ u(x, y) = (1 - 60\pi^6 x^2 + 294\pi^{12} t^4 - 252\pi^{18} t^6 + 33\pi^{24} t^8) \cos(\pi x) x - 4\pi^3 t(3 - 42\pi^6 t^2 + 84\pi^{12} t^4 - 30\pi^{18} t^6 + \pi^{24} t^8) \sin(\pi x) \]

The approximation can also be obtained by y-direction or by alternating use of x- and y-directions iterations formula.

Fig. 1. The numerical results for \( Y_4 \) in comparison with the analytic solution \( u(x, y) \) when \( t = 0.05 \) with initial condition of Eq.(21) by means of HPM.

Fig. 1.1. The plots of the numerical results for \( Y_4 \) in comparison with the analytic solution \( u(x, y) \) when \( t = 0.05 \) with initial condition of Eq.(21) by means of HPM.
Fig. 1. The numerical results for $Y_d$ in comparison with the analytic solution $u(x, y)$ when $t = 0.05$ with initial condition of Eq. (21) by means of ADM

IV. COMPARISON AMONG HPM, VIM AND ADM

It can be seen from the examples studied, that:

1. Comparison among HPM, VIM and ADM shows that although the results of these methods when applied to the Dispersive equation are the same, HPM does not require specific algorithms and complex calculations, such as ADM or construction of correction functionals using general
Lagrange multipliers, such as VIM and is much easier and more convenient than ADM and VIM.

2. HPM handles linear and nonlinear problems in a simple manner by deforming a difficult problem into a simple one. But in nonlinear problems, we encounter difficulties to calculate the so-called Adomian polynomials, when using ADM. Also, optimal identification of Lagrange multipliers via the variational theory can be difficult in VIM.

V. CONCLUSIONS
In this letter, we have successfully developed HPM and ADM to obtain the exact solutions of Dispersive equation. The results are then compared with those of VIM. It is apparently seen that these methods are very powerful and efficient techniques for solving different kinds of problems arising in various fields of science and engineering and present a rapid convergence for the solutions. The solutions obtained show that the results of these methods are in agreement but HPM is an easy and convenient one.

REFERENCES