A Complementary Slackness Theorem for Linear Fractional Programming Problem

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Abstract-- In this paper, the complementary slackness theorem for Seshan’s dual in linear fractional programming problem is proved. A numerical example is presented to demonstrate the result.

Index Term-- Duality, Linear Fractional Programming.

I. INTRODUCTION

In 1960’s and 1970’s several authors, Swarup[11], Bector[1],[2], Chadha [3], Kaska[13], Gol’stein [5],[6], Sharma and Swarup [10], Seshan [9] and many other authors proposed different type of dual problems related to the primal LFP problem consisting in maximizing and minimizing linear – fractional objective function subject to a system of linear constraints. Most of the authors proposed a dual form in which the objective function is linear. Some of them are based on the well known Charnes and Coopers transformation [14] and leads to the duality theory of linear programming. Only Sharma and Swarup [10] , Seshan [9] and Bector [1],[2] have defined a dual form in which the objective function is fractional, that is, ratio of two linear function. Jahan and Islam [12] showed that all these duals proposed by different authors are actually equivalent to one another. Most of the authors proved the duality theorems. Some of them did not proved Complementary Slackness Theorem. Gol’stein stated this theorem without proof. Also Sharma and Swarup , and Seshan are silent about the Complementary Slackness Theorem. In this paper complementary slackness theorem for Seshan’s dual is proved.

II. DUAL OF A LINEAR FRACTIONAL PROGRAM

In 1972 Swarup and Sharma [10] proposed a dual which has a special feature that both the problem (Primal and dual) are linear fractional. But they considered a primal problem in which constant term does not appear in both the numerator and denominator of the objective function.

In 1980 Seshan [9] extended their work to the general case where constant term has permitted to appear in both the numerator and denominator of the objective function and the constraints of the dual are also generalized. Consider the primal problem (PP)

\[( PP ) : \text{Maximize } f(x) = \frac{C(x)}{D(x)} = \frac{c'x + \alpha}{d'x + \beta} \quad (2.1)\]

subject to \[Ax \leq b \quad (2.2)\]
\[x \geq 0 \quad (2.3)\]

where \( d'x + \beta > 0 \), \( \forall x = (x_1, x_2, \ldots, x_n)' \in S \).

where \( S = \{ x : Ax \leq b , x \geq 0 \} \) is the feasible set which is assumed to be nonempty and bounded, and that \( f \) is not constant on \( S \).

\( A \) is a \( m \times n \) matrix,
\[x, c, d \in \mathbb{R}^n\]
\[b \in \mathbb{R}^m, \alpha, \beta \in \mathbb{R}\]
\[c', d'\] denotes transpose of vectors \( c \) and \( d \) respectively.

Seshan [9] proposed the following dual form for the primal problem (PP):

\[(\text{SeDP}): \text{Minimize } g(u, v) = \frac{c'u + \alpha}{d'u + \beta} \quad (2.4)\]

Subject to

\[cd'u - dc'u - A'v \leq \alpha d - \beta c \]
\[\alpha d'u - \beta c'u + b'v \leq 0\]
\[u \geq 0, \quad v \geq 0\]

Where \( u \in \mathbb{R}^n \) and \( v \in \mathbb{R}^m \).

This dual form can be written as follows

\[(\text{SeDP}): \text{Minimize } g(u, v) = \frac{c'u + \alpha}{d'u + \beta} \quad (2.4)\]

Subject to

\[A'v - cd'u + dc'u \geq \beta c - \alpha d \quad (2.6)\]
\[\beta c'u - \alpha d'u - b'v \geq 0\]
\[u \geq 0, \quad v \geq 0\]

Where \( u \in \mathbb{R}^n \) and \( v \in \mathbb{R}^m \).

Seshan proved the following theorems...
Theorem 1: (Weak Duality Theorem)
If $x$ is any feasible solution of (PP) and $(u, v)$ is a feasible solution of (SeDP), then $f(x) \leq g(u, v)$.

Theorem 2: (Direct Duality Theorem)
If $x^*$ solves (PP), then there exists $(u^*, v^*)$ which solves (SeDP) such that $f(x^*) = g(u^*, v^*)$.

Theorem 3: (Converse Duality Theorem)
If $(u^*, v^*)$ solves (SeDP), then there exists $x^*$ which solves (PP) such that $f(x^*) = g(u^*, v^*)$.

Seshan stated the following result without proof:

If $x$ is feasible solution of (PP) and $(u, v)$ is a feasible solution of (SeDP) such that $f(x) = g(u, v)$, then $x$ solves (PP) and $(u, v)$ solves (SeDP).

Which is the Optimality Criteria Theorem.

Seshan [9] is silent about the Complementary Slackness Theorem. Here we prove the complementary slackness theorem related to the dual proposed by Seshan[9].

III. COMPLEMENTARY SLACKNESS THEOREM
If $S$ and $W$ are slack and surplus column vectors associated with the primal problem (PP) and dual problem (SeDP) respectively, then $(x^*, s^*)$ solves the primal problem and $(u^*, v^*, w^*)$ solves the dual problem if and only if

$$w^* x^* + s^* v^* = 0$$

Or

$$x_j^* W_j^* = 0 \quad j = 1, 2, ..., n$$
$$v_i^* S_i^* = 0 \quad i = 1, 2, ..., m$$

Proof:
Let $S$ & $W$ be the slack vector and surplus vector of the primal problem (PP) and dual problem (SeDP) respectively. Then (2.2) implies:

$$Ax + s = b$$

$$\Rightarrow x^t A^t v + s^t v = b^t v$$

$$\Rightarrow x^t A^t v = b^t v - s^t v$$

where $s \geq 0$ (3.1)

Also from (2.6) we have,

$$A^t v - cd^t u + dc^t u - w = \beta c - \alpha d$$

$$\Rightarrow x^t A^t v - c^t xd^t u + d^t xc^t u - w^t x = \beta c^t x - \alpha d^t x$$

$$\Rightarrow x^t A^t v = \beta c^t x - \alpha d^t x + c^t xd^t u - d^t xc^t u + w^t x$$

where $w \geq 0$ (3.3)

From (2.7) we have, $b^t v \leq \beta c^t u - \alpha d^t u$

Using (3.1) and (3.3) it follows that

$$b^t v - s^t v = \beta c^t x - \alpha d^t x + c^t xd^t u - d^t xc^t u + w^t x$$

$$\Rightarrow c^t xd^t u + \beta c^t x - d^t xc^t u - \alpha d^t x + w^t x + s^t v = b^t v$$

$$\Rightarrow c^t x(d^t u + \beta) - d^t x(c^t u + \alpha) + w^t x + s^t v = b^t v$$

$$\leq \beta c^t u - \alpha d^t u$$

$$\Rightarrow c^t x(d^t u + \beta) - d^t x(c^t u + \alpha) + w^t x + s^t v - \beta c^t u + \alpha d^t u \leq 0$$

$$\Rightarrow (c^t x + \alpha)(d^t u + \beta) - (d^t x + \beta)(c^t u + \alpha) + w^t x + s^t v \leq 0$$

(3.5)

Now, if $(x^*, s^*)$ solves the primal problem (PP) and $(u^*, v^*, w^*)$ solves the dual problem (SeDP) then by direct duality theorem of Linear Fractional Program:

$$\frac{c^t x^* + \alpha}{d^t x^* + \beta} = \frac{c^t u^* + \alpha}{d^t u^* + \beta}$$

$$\Rightarrow (c^t x^* + \alpha)(d^t u^* + \beta) = (d^t x^* + \beta)(c^t u^* + \alpha)$$

(3.6)

Thus (3.5) and (3.6) implies, at an optima

$$w^* x^* + s^* v^* \leq 0$$

(3.7)

Since $w^* \geq 0, x^* \geq 0, s^* \geq 0, v^* \geq 0$

It follows that $w^* x^* + s^* v^* \geq 0$ (3.8)

From (3.7) and (3.8) it can be conclude that

$$w^* x^* + s^* v^* = 0$$

or

$$x_j^* W_j^* = 0 \quad j = 1, 2, ..., n$$
$$v_i^* S_i^* = 0 \quad i = 1, 2, ..., m$$

Conversely, let $w^* x^* + s^* v^* = 0$

$$\Rightarrow x^t A^t v - c^t xd^t u + dc^t u - w = \beta c^t - \alpha d$$

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To prove the converse, Dinkelbach [4] parametric approach has been used.

Dinkelbach [4] by defining $\lambda^* = \frac{c^t u^* + \alpha}{d^t u^* + \beta}$ considered the following linear programming problem:

**DDP:** Minimize $(c^t u + \alpha) - \lambda (d^t u + \beta)$  \hspace{1cm} (3.9)

Subject to

$$
\begin{align*}
A'v - cd' u + dc' u & \geq \beta c - \alpha d \\
\beta c' u - \alpha d' u - b'v & \geq 0 \\
u & \geq 0, \quad v \geq 0
\end{align*}
$$

Which can be written as:

**DDP:** Minimize $(c - \lambda d)^t u + \alpha - \lambda \beta$  \hspace{1cm} (3.10)

Subject to

$$
\begin{align*}
(dc' - cd') u + A'v & \geq \beta c - \alpha d \\
(\beta c' - \alpha d') u - b'v & \geq 0 \\
u & \geq 0, \quad v \geq 0
\end{align*}
$$

Dinkelbach [4] has proved that if $(u^*, v^*)$ solves (SeDP) then $(u^*, v^*)$ also solves (DDP) and the optimal value of the objective function of (DDP) is 0.

Now, the Dual of the linear programming problem (DDP) is

**D1:** Maximize $L(z, \mu) = (\beta c - \alpha d)^t z + \alpha - \lambda \beta$

subject to

$$
\begin{align*}
(d c' - c d')^t z + (\beta c - \alpha d) \mu & \leq c - \lambda d \\
A z - b \mu & \leq 0 \\
z & \geq 0, \quad \mu \geq 0
\end{align*}
$$

Where $x, z \in \mathbb{R}^n$ and $\mu \in \mathbb{R}$

Which implies

**D1:** Maximize $L(z, \mu) = (\beta c' - \alpha d')^t z + \alpha - \lambda \beta$

subject to

$$
\begin{align*}
(c d' - d c')^t z + (\beta c - \alpha d) \mu & \leq c - \lambda d \\
A z - b \mu & \leq 0 \\
z & \geq 0, \quad \mu \geq 0
\end{align*}
$$

Where $z \in \mathbb{R}^n$ and $\mu \in \mathbb{R}$

By Direct Duality Theorem of linear programming

$L(z^*, \mu^*) = (\beta c' - \alpha d')^t z^* + \alpha - \lambda^* \beta = 0$  \hspace{1cm} (3.17)

And $\mu^* \neq 0$. For if $\mu^* = 0$, then form (3.15)

$$
A z^* \leq 0, \quad z^* \geq 0
$$

Since $S = \{x: Ax \leq b\}$, the feasible set which has been assumed to be nonempty and bounded, there exists an $x$ on $S$ such that

$Ax \leq b, \quad x \geq 0$.

Then $A(x + t z^*) \leq b, \quad x + t z^* \geq 0$, for every $t > 0$.

Hence $x + t z^* \geq 0$ is in $S$ for every $t > 0$. Which is contradiction if $z^* \neq 0$ since $S$ is bounded.

If both $\mu^* = 0$ and $z^* = 0$, then from (3.17)

$\alpha - \lambda^* \beta = 0$. From (3.14) we get $c - \lambda^* d \geq 0$.

If $x$ is any feasible solution, then $c^t x \geq \lambda^* d^t x$ and

$\alpha = \lambda^* \beta$. Hence $c^t x + \alpha \geq \lambda^* (d^t x + \beta)$

$$
\Rightarrow \frac{c^t x + \alpha}{d^t x + \beta} \geq \lambda^*
$$

$$
\Rightarrow f(x) \geq \lambda^* = g(u^*, v^*)
$$

But by Weak Duality Theorem of linear fractional programming

$f(x) \leq g(u^*, v^*) = \lambda^*$

Therefore $f(x) = \lambda^*$, for every feasible solution $x$ of (PP) which implies that $f$ is constant on $S$, a contradiction to assumption.

Therefore $\mu^* > 0$.

Applying Complementary Slackness Theorem for linear programming on (DDP) and (D1)

(3.11)$\Rightarrow$

$$
A^t v^* - c^t d^t u^* + d^t z^* c^t u^* - \beta c^t z^* + \alpha d^t z^* = 0
$$

(3.12)$\Rightarrow$

$$
\beta^* c^t u^* - \alpha^* d^t u^* - \mu^* b^t v^* = 0
$$

(3.15)$\Rightarrow$

$$
A^t v^* - \mu^* b^t v^* = 0
$$

Now (3.18) + (3.19) - (3.20) it follows that

$$
-c^t z^* d^t u^* + d^t z^* c^t u^* - \beta c^t z^* + \alpha d^t z^* + \beta^* c^t u^* - \alpha^* d^t u^* = 0
$$

$$
\Rightarrow -(c^t z^* (d^t u^* + \beta) + d^t z^* (c^t u^* + \alpha) + \beta^* c^t u^* - \alpha^* d^t u^* = 0
$$

$$
\Rightarrow -(c^t z^* + \alpha^* \beta^*) (d^t u^* + \beta) + (d^t z^* + \beta^* \alpha^*) (c^t u^* + \alpha) = 0
$$
=> (d′z′ + βu*)((c′u* + α) = (c′z′ + αμ)(d′u* + β)

=> μ*(d′z′ + β)(c′u* + α) = μ*(c′z′ + αμ)(d′u* + β)

(3.21)

Since μ* > 0, setting x* = z′/μ, (3.21) implies

(d′x′ + β)(c′u* + α) = (c′x′ + α)(d′u* + β)

=> c′x′ + α = d′x′ + β

Hence the theorem is proved.

To verify the theorem, a numerical example is considered in the next section. Using Complementary Slackness Theorem the solution of the primal problem is obtained from the solution of the associated dual problem proposed by Seshan [9].

IV. NUMERICAL EXAMPLE

Consider the primal problem

(NPP): Maximize f(x) = 3x1 + x2 + x3 + 1/x1 + 2x2 + 3x3 + 2

subject to

2x1 + x2 + 3x3 ≤ 4
x1 + 2x2 + x3 ≤ 1
xj ≥ 0, j = 1, 2, 3

Here, c = [3 1 1], d = [1 2 3], A = [2 1 3], b = [4 1], α = 1, β = 2

Seshan’s Dual Problem:

Seshan’s dual of the above primal problem (NPP) is as follows:

Here  g(u, v) = c′u + α d′u + β

And  cd′u – dc′u – A′v ≤ αd – βc

=> 1(1 2 3 u1 2 3 1 u2 1 2 3 1 u3) – 2(1 3 1 1 u1 3 1 1 u2 1 1 u3) v1 – 3 1 1 v2 ≤ 0

=> 5u2 + 8u3 – 2v1 – v2 ≤ –5

Thus Seshan’s dual to primal problem is

(SeDP): Minimize g(u, v) = 3u1 + u2 + u3 + 1/u1 + 2u2 + 3u3 + 2

subject to

5u2 + 8u3 – 2v1 – v2 ≤ –5

-5u1 + u3 – v1 – 2v2 ≤ 0

-8u1 – u2 – 3v1 – v2 ≤ 1

-5u1 + u2 + 4v1 + v2 ≤ 0

u1 ≥ 0, j = 1, 2, 3

and

v1 ≥ 0, i = 1, 2

Which can be written as

(SeDP): Minimize g(u, v) = 3u1 + u2 + u3 + 1/u1 + 2u2 + 3u3 + 2

subject to

-5u1 + u3 – v1 – 2v2 ≤ 0

-8u1 – u2 – 3v1 – v2 ≤ 1

-5u1 + u2 + 4v1 + v2 ≤ 0

u1 ≥ 0, j = 1, 2, 3

and

v1 ≥ 0, i = 1, 2
If \( w_1 \) is the surplus and \( w_2, w_3 \) are slack variables corresponding to constraints (4.2), (4.3) and (4.4) respectively.

Solving this linear fractional programming problem (SeDP) by Martos's [7,8] Simplex type method we get

\[
\begin{align*}
 u_1^* &= 1, u_2^* = 0, u_3^* = 0, v_1^* = 0, v_2^* = 5, \\
 w_1^* &= 0, w_2^* = 15, w_3^* = 14 \\
 g(u^*, v^*) &= \frac{3.1 + 0 + 0 + 1}{1 + 2.0 + 3.0 + 2} = \frac{4}{3}
\end{align*}
\]

Introducing slack variables \( s_1 \) and \( s_2 \) in (NPP) the standard form of the primal problem becomes:

Maximize \( f(x) = \frac{3x_1 + x_2 + x_3 + 1}{x_1 + 2x_2 + 3x_3 + 2} \)

subject to

\[
\begin{align*}
 2x_1 + x_2 + 3x_3 + s_1 &= 4 \quad (4.6) \\
 x_1 + 2x_2 + x_3 + s_2 &= 1 \quad (4.7) \\
 x_j &\geq 0, \quad j = 1, 2, 3 \\
 s_i &\geq 0, \quad i = 1, 2
\end{align*}
\]

Now, by complementary slackness theorem

\[
\begin{align*}
 w^* x^* + s^* v^* &= 0 \\
 x_j^* w_j^* &= 0 \quad j = 1, 2, \ldots, n \\
 v_i^* s_i^* &= 0 \quad i = 1, 2, \ldots, m
\end{align*}
\]

So \( v_2^* = 5 > 0 \Rightarrow s_2^* = 0 \)

\( w_2^* = 15 > 0 \Rightarrow x_2^* = 0 \)

\( w_3^* = 14 > 0 \Rightarrow x_3^* = 0 \)

Substituting the values of \( x_2^* \), \( x_3^* \) and \( s_2^* \) in (4.6) and (4.7)

\[
\begin{align*}
 (4.6) \Rightarrow 2x_1^* + s_1^* &= 4 \\
 (4.7) \Rightarrow x_1^* &= 1
\end{align*}
\]

Solving we get \( x_1^* = 1, s_1^* = 2 \)

Hence the value of the objective function is

\[
f(x^*) = \frac{3.1 + 0 + 0 + 1}{1 + 2.0 + 3.0 + 2} = \frac{4}{3}
\]

So Complementary Slackness Theorem for Seshan’s [9] Dual, which is proved in section III, is verified.

### V. Some Remarks

**Remark 1:** Swarup and Sharma [10] have considered a primal problem in which constant term does not appear in both the numerator and denominator of the objective function. They proposed a linear fractional programming problem as a dual of the primal problem. In 1980 Seshan [9] extended their work to the general case where constant term has permitted to appear in both the numerator and denominator of the objective function of the primal problem and the constraints of the dual are also generalized. But the practical usability of the vectors \( u \) and \( v \) used to define the dual problem of Swarup and Sharma and therefore of Seshan is still an open question.

**Remark 2:** The dual (DSeDP) of the dual (SeDP) (2.4)-(2.5)

\[
(DSeDP): \text{Maximize } g(x) = \frac{c^t x + \alpha}{d^t x + \beta}
\]

Subject to

\[
\begin{align*}
 (cd^t - dc^t) x - (dc^t - cd^t) z &\geq (\alpha d - \beta c) (1 - \lambda) \\
 (\alpha d^t - \beta c^t) x + (\beta c^t - \alpha d^t) z &\geq 0 \\
 A x - b \lambda &\leq 0 \\
 x &\geq 0, \quad z \geq 0, \quad \lambda \geq 0
\end{align*}
\]

Where \( x, z \in \mathbb{R}^n \) and \( \lambda \in \mathbb{R} \)

After manipulation the above becomes:

\[
(DSeDP): \text{Maximize } g(x) = \frac{c^t x + \alpha}{d^t x + \beta}
\]

Subject to

\[
\begin{align*}
 (cd^t - dc^t) x - (cd^t - dc^t) z &\geq (\alpha d - \beta c) (1 - \lambda) \\
 (\alpha d^t - \beta c^t) x + (\beta c^t - \alpha d^t) z &\geq 0 \\
 A x - b \lambda &\leq 0 \\
 x &\geq 0, \quad z \geq 0, \quad \lambda \geq 0
\end{align*}
\]

Where \( x, z \in \mathbb{R}^n \) and \( \lambda \in \mathbb{R} \)

Which implies

\[
(DSeDP): \text{Maximize } g(x) = \frac{c^t x + \alpha}{d^t x + \beta}
\]

Subject to

\[
\begin{align*}
 (cd^t - dc^t) z - (cd^t - dc^t) x &\leq (\alpha d - \beta c) (1 - \lambda) \\
 (\alpha d^t - \beta c^t) z + (\beta c^t - \alpha d^t) x &\leq 0 \\
 A x + b \lambda &\leq 0 \\
 x &\geq 0, \quad z \geq 0, \quad \lambda \geq 0
\end{align*}
\]
Where $\mathbf{x}, \mathbf{z} \in \mathbb{R}^n$ and $\mathbf{\lambda} \in \mathbb{R}^m$

Any feasible solution $\mathbf{x}$ of (PP) gives rise to a feasible solution of (DSeDP) if we take $z = x$ and $\mathbf{\lambda} = 1$. Further the objective function values are equal. But the converse is not true. Hence dual of (SeDP) (2.4) – (2.5) is not equivalent to (PP) (2.1) - (2.3).

REFERENCES


