Abstract-- In this paper, we present a concept of conditionally exponential convex functions and we prove that under suitable assumptions it is possible to define a Dirichlet forms constructed by a class of pseudo differential operators with symbols defined in terms of conditionally exponential convex functions.

Index Term-- Pseudo differential operators –Dirichlet forms- conditionally exponential convex function.

1- CONDITIONALLY EXPONENTIAL CONVEX FUNCTIONS.

Definition 1.1. A real valued function $\Psi : \mathbb{R}^n \rightarrow \mathbb{R}$ is said to be conditionally exponential convex if for any $x_1, \ldots, x_n \in \mathbb{R}^n$ and $c_1, \ldots, c_n \in \mathbb{R}$ we have

$$\sum_{j,k=1}^{n} [\psi(x_j) + \psi(x_k) - \psi(x_j + x_k)]c_j c_k \geq 0$$

(1.1)

Theorem 1.1.

([8] Theorem 3.7). A continuous function $\Psi : \mathbb{R}^n \rightarrow \mathbb{R}$ is conditionally exponential convex on $\mathbb{R}^n$ if and only if it can be represented in the form:

$$\psi(\xi) = C - \zeta(\xi) + \int_{\mathbb{R}^n \setminus \{0\}} \left[1 - \exp(x, \xi) + \frac{(x; \xi)^2}{1 + \|x\|^2} \right] \|x\|^2 d\mu(x)$$

(1.2)

where $C \geq 0$ is a constant, $\zeta : \mathbb{R}^n \rightarrow \mathbb{R}$ is a continuous negative quadratic form on $\mathbb{R}^n$ and $\mu$ is a positive bounded measure on $\mathbb{R}^n \setminus \{0\}$. Moreover, the triple $(C; \zeta, \mu)$ is uniquely determined by the function $\psi$.

Lemma 1.1.

Let $\psi, \psi_1, \psi_2$ be continuous conditionally exponential convex functions. Then

(i) $\psi(\xi) \geq \psi(0) \geq 0; \psi(-\xi) = \psi(\xi)$ ($\xi \in \mathbb{R}^n$)

(ii) $\psi(\xi) \leq C_\psi (1 + |\xi|^2)$ for some constant $C_\psi \geq 0$.

(iii) $C_1 \psi_1 + C_2 \psi_2$ is a continuous conditionally exponential convex function.

For the proof see [1,2,3].

(iv) $\psi_1^{1/2}$ is a continuous conditionally convex function and the inequality

$$|\psi_1^{1/2}(\xi) - \psi_1^{1/2}(\eta)| \leq \psi_1^{1/2}(\xi + \eta)$$

holds for all $\xi, \eta \in \mathbb{R}^n$.

For the proof of (iv) we have the following:

Lemma 1.2.

Let $\Psi^2 : \mathbb{R}^n \rightarrow \mathbb{R}$ be a continuous conditionally exponential convex function and $\psi$ be its square root then for $\xi, \eta \in \mathbb{R}^n$ we have

$$|\psi(\xi) - \psi(\eta)| \leq \psi(\xi + \eta)$$

(1.3)

Proof.

By definition of conditionally exponential convex function we have

$$\det\begin{pmatrix} \psi^2(\xi) + \psi^2(\xi) - \psi^2(0) & \psi^2(\xi) + \psi^2(\eta) - \psi^2(\xi + \eta) \\ \psi^2(\eta) + \psi^2(\xi) - \psi^2(\eta + \xi) & \psi^2(\eta) + \psi^2(\eta) - \psi^2(0) \end{pmatrix} \geq 0$$

And therefore by (i) and (ii)

$$2\psi^2(\xi) - 2\psi^2(\eta) \geq (2\psi^2(\xi) - \psi^2(0))(2\psi^2(\eta) - \psi^2(0)) \geq (\psi^2(\xi) + \psi^2(\eta) - \psi^2(\xi + \eta))^2$$

Or

$$2\psi(\xi)\psi(\eta) \geq \psi^2(\xi) + \psi^2(\eta) - \psi^2(\xi + \eta)$$

Thus we find

$$(\psi(\xi) - \psi(\eta))^2 = \psi^2(\xi) - 2\psi(\xi)\psi(\eta) + \psi^2(\eta) \leq \psi^2(\xi + \eta)$$
$$H^{1/2,\nu}(R^n) = \left\{ u \in L^2(R^n) : \int_{R^n} (1 + \psi(\xi))|\widehat{u}(\xi)|^2 d\xi < \infty \right\}$$

Where \( \widehat{u}(\xi) = \int e^{-ix \cdot \xi} u(x) dx \)

on \( H^{1/2,\nu}(R^n) \) we have the norm

$$\| u \|^2_{1/2,\nu} = \int_{R^n} (1 + \psi(\xi))|\widehat{u}(\xi)|^2 d\xi$$

For \( \psi(\xi) = |\xi|^{2\nu} \) we obtain the usual sobolev \( v \) spaces. For more details see [4,5], with this norm the space \( H^{1/2,\nu}(R^n) \)

is a Hilbert space and \( C_0^\infty(R^n) \) is a dense subspace of \( H^{1/2,\nu}(R^n) \) for all \( u \in H^{1/2,\nu}(R^n) \).

$$\| u \|^2_{1/2,\nu} = \| u \|^2_0 + \| \psi^{1/2}(D)u \|^2_0$$

Using Lemma 1.1, we can construct the chain

$$H^1(R^n) \subset H^{1/2,\nu}(R^n) \subset L^2(R^n)$$

(1.5)

**Proposition 1.1.**

Suppose \( \psi_1 \) and \( \psi_2 \) are two continuous conditionally exponential convex function and that

$$\psi_2(\xi) \leq \psi_1(\xi)$$

(1.6)

Holds for all \( \xi \in R^n \). Then it follows that

$$\| u \|^2_{1/2,\nu_2} \leq C \| u \|^2_{1/2,\nu_1} \psi_1$$

(1.7)

For all \( u \in H^{1/2,\nu}(R^n) \).

In particular \( H^{1/2,\nu_1}(R^n) \) is continuously embedded into the space \( H^{1/2,\nu_2} \). Since the function, \( |\xi|^2 \) is a continuous conditionally convex function then for any \( t \in (0,2] \) it follows that

$$C_t (1 + |\xi|^2)^{1/2} \leq 1 + \psi(\xi), \quad t \in (0,2]$$

(1.8)

**Definition 1.2.**

A Dirichlet form on \( L^2(R^n) \) is a closed symmetric non-negative bilinear form \( B \) with domain \( D(B) \) such that

\( u \in D(B) \) implies \( u^+ \land 1 \in D(B) \) and

\( B(u^+ \land 1, u^+ \land 1) \leq B(u,u) \)

The pair \( (D(B),B) \) is called Dirichlet space [9,10,11]

2. A CLASS OF CLOSED BILINEAR FORMS

Let \( m \in N; n_j \in N \) for \( 1 \leq j \leq m \), let

$$\Psi_j : R^{n_j} \rightarrow R$$

be a continuous conditionally exponential convex function

$$\psi_j(\xi) = \int (1 - \cosh(x_j \xi)) \bar{\mu}_j dx_j$$

(2.1)

\( \bar{\xi}_j = \bar{\xi}_{nj} \in R^{n_j} \), where \( \bar{\mu}_j \) is a positive finite symmetric measure on \( R^{n_j} \), \( \setminus \{0\} \).

We denote by \( n = n_1 + n_2 + \cdots + n_m \) the image of \( \bar{\mu}_j \) under the mapping \( T_j : R^{n_j} \rightarrow R^n; x_j \rightarrow (0,\cdots,0,x_j,0,\cdots,0) \) is denoted by \( \mu_j \). Let \( b_j \in L^\infty(R^n) \) be independent of \( x_j \). We denote by \( x' = (x_1, \ldots, x_{j-1}; x_{j+1}, \ldots, x_m) \), \( x_k = x_{jk}, 1 \leq k \leq m \), and we identify \( R_j = R^{n_j} \times \cdots \times R^{n_j-1} \times R^{n_j} \times \cdots \times R^{n_j} \) with a subspace of \( R^n; b_j = b_j(x'_j) \). Let \( \Psi : R^n \rightarrow R \) be continuous conditionally exponential convex function

$$\psi(\xi) = \sum_{j=1}^m \psi_j(\xi_j)$$

in this case

$$B_{1/2,\nu} (R^n) = \left\{ u \in L^2(R^n) : \int_{R^n} (1 + \psi(\gamma))|\widehat{u}(\gamma)|^2 d\gamma < \infty \right\}$$

And \( \left( B_{1/2,\nu}(R^n);(\cdots)_{1/2,\nu} \right) \) is a Dirichlet space. For \( u \in H^{1/2,\nu}(R^n) \) the function \( (u^+ \land 1) \in H^{1/2,\nu}(R^n) \). For \( \phi \in C_0^\infty(R^n) \) we define

$$L\phi(x) = \sum_{j=1}^m b_j(x'_j) A_j \phi(x),$$

Where \( 1 \leq j \leq m \),

$$A_j \phi(x) = \int e^{\bar{x}_j \xi} \psi_j(\xi_j) \hat{\phi}(\xi_j) d\xi_j$$

Where \( \hat{\phi}(\xi) = \int e^{-\bar{x}_j \xi} \phi(x) dx \) we observe
\[ L \phi(x) = \int_{\mathbb{R}^n} e^{i \xi \cdot x} \left[ \sum_{j=1}^{m} b_j(x', y_j(x')) \right] \hat{\phi}(\xi) d\xi \]

Since \( b_j \) is independent of \( x \), we can associate with \( L \) a symmetric bilinear form defined on \( C_0^\infty(\mathbb{R}^n) \) by

\[ B(\phi, \nu) = (L \phi, \nu)_0 = \sum_{j=1}^{m} \int_{\mathbb{R}^n} b_j(x') A_j^{1/2} \phi(x) \overline{A_j^{1/2} \nu(x)} dx, \]

where \( A_j^{1/2} \) has the symbol \( a_j^{1/2}, 1 \leq j \leq m. \)

**Proposition 2.1.**

For all \( \phi, \nu \in C_0^\infty(\mathbb{R}^n) \) we have

\[ |B(\phi, \nu)| \leq C \|\phi\|_{1/2, \nu}\|\nu\|_{1/2, \nu}, \]

which gives (2.2).

**Proof.**

Since \( C_0^\infty(\mathbb{R}^n) \) is dense in \( H^{1/2, \nu}(\mathbb{R}^n) \) it is sufficient to prove (2.2) for all test functions. For \( \phi, \nu \in C_0^\infty(\mathbb{R}^n) \) it follows that

\[ |B(\phi, \nu)| = \left| \sum_{j=1}^{m} (b_j(x') A_j^{1/2} \phi(x); A_j^{1/2} \nu(x))_0 \right| \leq C \sum_{j=1}^{m} \|A_j^{1/2} \phi\|_0 \|A_j^{1/2} \nu\|_0, \]

where \( C \|A_j^{1/2} \phi\|_0 \leq C \|\phi\|_{1/2, \nu} \) for all \( \phi \in C_0^\infty(\mathbb{R}^n) \) it follows that \( B \) can be regarded as a continuous bilinear form on \( H^{1/2, \nu}(\mathbb{R}^n) \).

**Proposition 2.2.**

Let \( B \) as in Proposition 2.1 and suppose that there exists a constant \( d_0 > 0 \) such that

\[ b_j(x') \geq d_0 \quad \text{for all } j = 1, \ldots, n \]

(2.3)

Then we have for all \( \phi \in H^{1/2, \nu}(\mathbb{R}^n) \)

\[ B(\phi, \phi) \geq 0 \]

(2.4)

And

\[ B(\phi, \phi) \geq d_0 \|\phi\|^2_{1/2, \nu} - d_0 \|\phi\|^2_0 \]

(2.5)

**Proof.**

For \( \phi \in H^{1/2, \nu}(\mathbb{R}^n) \), we have

\[ B(\phi, \phi) = \sum_{j=1}^{m} \left( b_j(x') A_j^{1/2} \phi, A_j^{1/2} \phi \right)_0 = \sum_{j=1}^{m} \int_{\mathbb{R}^n} b_j(x') \left| A_j^{1/2} \phi(x) \right|^2 dx \]

\[ \geq d_0 \sum_{j=1}^{m} \int_{\mathbb{R}^n} \left| A_j^{1/2} \phi(x) \right|^2 dx \geq 0 \]

(2.6)

Which gives (2.4)

Note that (2.5) is a Carding-inequality [9,10] and we get by (2.6) and (2.3)

\[ B(\phi, \phi) \geq d_0 \int_{\mathbb{R}^n} \left( 1 + \sum_{j=1}^{m} \nu_j(\xi_j) \right) \left| \tilde{\phi}(\xi) \right|^2 d\xi - d_0 \int_{\mathbb{R}^n} \left| \phi(\xi) \right|^2 d\xi \]

\[ \geq d_0 \|\phi\|^2_{1/2, \nu} - d_0 \|\phi\|^2_0 \]

From proposition 2.1 and 2.2 it follows that \( B \) with domain \( D(B) = H^{1/2, \nu}(\mathbb{R}^n) \) is a closed symmetric bilinear form on \( L^2(\mathbb{R}^n) \).

**Theorem 2.1.**

Let \( B \) as in Proposition 2.2 For \( \phi, \nu \in H^{1/2, \nu}(\mathbb{R}^n) \) we have

\[ B(\phi, \nu) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} (\phi(x+y) - \phi(x)) (\nu(x+y) - \nu(x)) J(x, dy) dx \]

(2.7)

Where

\[ J(x, dy) = \frac{1}{2} \sum_{j=1}^{m} b_j(x') \mu_j(dy) \]

(2.8)

**Proof.**

Using the notations introduced above, (2.1), we write

\[ \phi(x) = \phi(x', x_j') \]

And

\[ (F_{j'} \phi)(\xi_j, x_j') = \int_{\mathbb{R}} e^{-|x_j'|^2/2} \phi(x_j', x_j') dx_j \]

(2.9)

For \( \phi \in C_0^\infty(\mathbb{R}^n) \) or more generally for \( \phi \in L_2(\mathbb{R}^n) \) we find
\[ B(\phi, \nu) = \sum_{j=1}^{n} \int_{R^n} b_j(x_j) A_j^{1/2} \phi(x) A_j^{1/2} \nu(x) dx \]

\[ = \sum_{j=1}^{m} \int_{R^{n-j}} b_j(x_j') \int_{R} A_j^{1/2} \phi(x) A_j^{1/2} \nu(x) dx_j d\xi_j \]

\[ = \sum_{j=1}^{n} \int_{R^{n-j}} b_j(\xi_j) (F_j \phi)(\xi_j, \tilde{x}_j) (F_j \nu)(\xi_j, \tilde{x}_j) d\xi_j d\tilde{x}_j \]

\[ = \sum_{j=1}^{n} b_j(\tilde{x}_j) I_j(\tilde{x}_j) d\tilde{x}_j \]

Where

\[ I_j(\tilde{x}_j) = \int_{R} A_j(\xi_j) (F_j \phi)(\xi_j, \tilde{x}_j) (F_j \nu)(\xi_j, \tilde{x}_j) d\xi_j \]

\[ \int R f(t) g(-t) = \int R R R R e^{(x+y)} f(x) g(x) dx dy dz \]

It follows that

\[ I_j(\tilde{x}_j) = \int_{R} \left( 1 - \cosh(y_j, \xi_j) \right) (F_j \phi)(\xi_j, \tilde{x}_j) (F_j \nu)(\xi_j, \tilde{x}_j) \tilde{\mu}_j(dy_j) d\xi_j \]

\[ I_j(\tilde{x}_j) = \frac{1}{2} \int_{R} \left( e^{\gamma_j(s_j)} - 1 \right) (F_j \phi)(\xi_j, \tilde{x}_j) (F_j \nu)(\xi_j, \tilde{x}_j) d\xi_j \tilde{\mu}_j(dy_j) \]

\[ = \frac{1}{2} \int_{R} \left( e^{\gamma_j(s_j)} - 1 \right) (F_j \phi)(\xi_j, \tilde{x}_j) e^{\gamma_j(s_j)} - 1 \]

\[ (F_j \nu)(\tau_j, \tilde{x}_j) d\sigma_j d\tau_j \tilde{\mu}_j(dy_j) dx_j \]

\[ = \frac{1}{2} \int_{R} \left( \phi(x_j + y_j, \tilde{x}_j) - \phi(x_j, \tilde{x}_j) \right) (\nu(x_j + y_j, \tilde{x}_j) - \nu(x_j, \tilde{x}_j) \tilde{\mu}_j(dy_j) d\tilde{x}_j \]

\[ = \frac{1}{2} \int_{R} \left( \phi(x + y) - \phi(x) \right) (\nu(x + y) - \nu(x)) \mu_j(dy) dx_j \]

Which finally gives

\[ B(\phi, \nu) = \frac{1}{2} \sum_{j=1}^{n} \int_{R^{n-j}} b_j(\tilde{x}_j) \int_{R} \left( \phi(x + y) - \phi(x) \right) (\nu(x + y) - \nu(x)) \mu_j(dy) dx_j d\tilde{x}_j \]

\[ = \int_{R^{n-j}} \left( \phi(x + y) - \phi(x) \right) (\nu(x + y) - \nu(x)) d\tilde{x}_j \]

\[ \frac{1}{2} \sum_{j=1}^{n} b_j(\tilde{x}_j) \mu_j(dy) dx_j \]

And therefore the theorem is proved. Now it is easily seen that

**Theorem 2.2.**

Suppose that \( B \) satisfies the assumptions of Proposition 2.2 and Theorem 2.1. Then \( (H^{1/2, \nu}(R^n); B) \) is a Dirichlet space.

**Proof.**

If \( n_1 = n_2 = \cdots = n_m = 1 \) then we obtain the result From [6,7].

**REFERENCES**


